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Learning trajectory based on fractional sub-constructs: Using fractions as quotients to introduce fractions

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Abstract: *Learning emphasizing fractions as a part-whole concept causes several limitations in developing fraction knowledge and inhibits proportional reasoning. We use fractions as quotients as the first context introduced in our learning trajectory. We report the teaching experiment results using the improved learning trajectory on thirty 4th grade students in Jakarta's public schools. The findings of this study indicate that the fractions as quotients used as the first stage in the learning trajectory can lay a solid foundation for the concept of partitioning in a variety of strategies and the concept of fractional parts. Besides, the developed learning trajectory has provided opportunities for students to learn about fractional mental operations, which are interrelated and serve as the basis for the development of proportional reasoning.*

Keywords: *Fractions subconstructs, partitioning, iterating, unitizing, multiplicative reasoning, proportional reasoning*

INTRODUCTION

Fractions are one of the concepts used to solve problems and communicate in everyday life, as not all daily life problems can be translated or solved with integers. For example, when purchasing sugar, sometimes it requires less than a kilo, or when slicing bread, it needs its parts to be shared with some people. Besides, the concept of fractions is essential for students to understand advanced mathematical concepts, such as arithmetic, algebra, geometry and measurement, probability, and statistics (Purnomo et al., 2019, 2017). However, many existing studies in the broader literature have indicated that fractions are one of the problematic mathematical content for students, especially at the elementary school level (Charalambous & Pitta-Pantazi, 2007; Purnomo et al., 2019, 2017). The possible reason is that instrumental teaching and learning are still implemented in mathematics classrooms (Purnomo et al., 2014).

As the importance of fractions in daily life is generally known, the curriculum structure and contents should be organized well to help students understand and apply fractions. As a matter of fact, In Indonesian curricula, research-based learning trajectories to construct the concept of fractions have not yet been considered in the teaching and learning process and textbooks (Rahmawati et al., 2020). Clements and colleagues (Clements et al., 2019, 2020) argue that research-based learning trajectories are essential for developing instructional planning, curriculum (textbooks), teaching, and assessment. Therefore, our research focuses on developing a learning trajectory based on fractions subconstructs to promote fractions concept development.

Fractions Subconstructs: Meanings of Fractions

Kieren (1976) introduced interrelated meanings or subconstructs of fractions, namely fractions as ratios, fractions as measures, fractions as division (also known as fractions as quotients), and fractions as operators. Behr and colleagues (Behr et al., 1983) added another subconstruct, fractions as part-whole (Lamon, 2007). They established theoretical relationships among the five subconstructs, the basic operations of fractions, fraction equivalence, and problem-solving (Charalambous & Pitta-Pantazi, 2007). The summary of the five subconstructs is presented in Table 1 (Clarke et al., 2011; Lamon, 2007, 2012; Purnomo, 2015; Watanabe et al., 2017; Wilkins & Norton, 2018).

Subconstructs	**Descriptions**	**Mathematical Statement**	**Example**
Fractions as part-whole	A situation in which a continuous quantity or a set of discrete objects are partitioned into equal parts	$\frac{m}{n}$ whereas m is taken from n with fair sharing.	$\frac{2}{3}$ means two parts out of three equal parts of the pizza.

Fractions as quotient	Any fraction can be seen as the result of a division situation.	$\frac{m}{n}$ as dividing a quantity m into n with fair sharing.	$\frac{2}{3}$ means two pizzas divided by three people or each person's amount when three people share a 2-unit of pizza.
Fractions as measures	Fractions can represent a measure of the quantity relative to one unit of that quantity	$\frac{m}{n}$, as being m measures of the unit fraction, namely $\frac{1}{n}$, or m is iterations of $\frac{1}{n}$.	Two pieces of $\frac{1}{3}$-pizzas are vegetables.
Fractions as operator	Fraction as functions apply to some number, object, or set	$\frac{m}{n}$, whereas m is applied to the number of $\frac{1}{n}$.	$\frac{2}{3}$ of 24 marbles or $\frac{2}{3}$ of glass or $\frac{2}{3}$ kg of sugar.
Fractions as ratio	Fractions as ratios refer to the comparison between two quantities; therefore, it is considered comparative index, rather than a number	$\frac{m}{n}$, whereas it is stated the relationship between m and n, whereas $m + n$ = whole (part-part relationship) or n = whole (part-whole relationship)	The case of two green marbles and six red marbles can be interpreted differently. The first case is the relationship of parts to other parts, 2:6 or 1:3 (as the number of green marbles is 1/3 of the number of red marbles or the number of red marbles is three times the number of marbles green). The second case is the relationship of parts to the whole, 2:8 or 1:4 (using the same analogy).

Table 1. The Five Subconstructs of Fractions

As shown in Table 1, the five subconstructs of fractions are interrelated. Each plays an essential role in developing the concept of fractions and other subjects that require the concept of fractions. Referring to the study conducted by Charalambous and Pitta-Pantazi (2007), the part-whole construct is a fundamental concept for developing other subconstructs. An illustration of the relationship is illustrated in Figure 1.

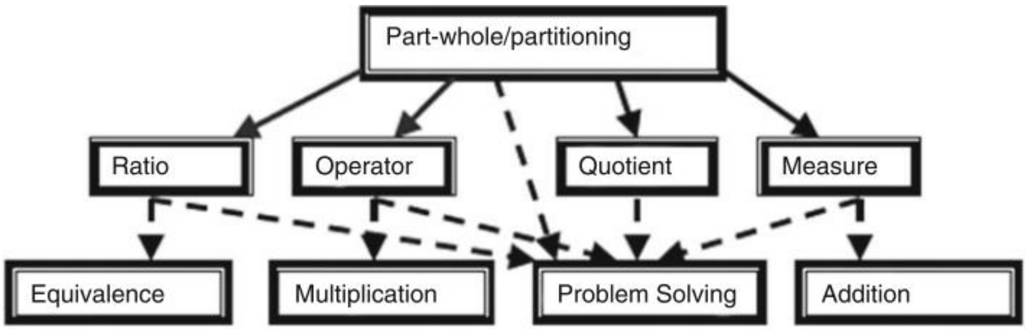

Figure 1. The five subconstructs of fractions and their relationships

Based on Figure 1, the part-whole subconstruct is at the core of other subconstructs' development. It is also empirically corroborated by the results of Charalambous and Pitta-Pantazi's (2007) study on elementary school students in Cyprus. However, the relationship between subconstructs remains unclear as there is no further explanation concerning directions that connect the rest subconstruct. It is also in line with Lamon (2007) and Tsai and Li (2017), who argued that the five subconstructs are interrelated. However, a further explanation of the empirical relationship has not been found in the literature. Although many authors have conducted studies, this problem is still insufficiently explored.

Fractional Mental Operations

Fractional mental operations refer to fractional mental actions that have been abstracted from experience to become available for use in various situations (McCloskey & Norton, 2009; Norton & McCloskey, 2008; Steffe, 2004; Steffe & Olive, 2010). In this study, we focus on partitioning, iterating, and unitizing.

Previous studies have shown that partitioning and iterating activities determine the foundation for developing fractions knowledge, fractional number sense, and fraction operations (Purnomo, 2015; Shin & Lee, 2018; Siebert & Gaskin, 2006; Wilkins & Norton, 2018). Partitioning is defined as dividing an object or several objects into disjoint and exhaustive parts (Lamon, 2012). Some used the term of equipartitioning to indicate that the part in question is the same size (c.f. Steffe & Olive, 2010). Meanwhile, iterating refers to a mental process that repeatedly copies specific fraction units to get the whole or other fraction units. Copying parts is repeatedly carried out to get a part of another whole, the whole itself, and more than the whole (Purnomo, 2015; Singh, 2000; Wilkins & Norton, 2018).

Partitioning activity is often identified with the introduction of fractions as part-whole relationships, i.e., determining the part that is the focus of observation of the whole. The whole unit can be discrete (e.g., six candies) or continuous (e.g., a tube). However, this activity can be developed through other subconstructs. An example of fractions as quotients is that when three children are sharing six cookies. It can be represented mathematically as $6 \div 3$ or 6/3, in which the process of sharing six cookies to three children can involve various possible partitioning strategies, such as distributive partitioning, halving, or recursive partitioning (Lamon, 1996; McCloskey & Norton, 2009; Shin & Lee, 2018; Steffe & Olive, 2010). This activity became an alternative to introduce fractions, where the part-whole subconstruct, which is put first in most curricula, was less successful in laying the foundations of the concept of fractions (Simon, Placa, Avitzur, & Kara, 2018; Watanabe, 2006; Wilkins & Norton, 2018).

Fractions as measures also accommodate partitioning and iterating activities. These partitions and iterations consist of mental activity that can be composed of one another and, more specifically, form inverses of each other to cancel each other out when compiled (Wilkins & Norton, 2018). However, some researchers note that the concept of fractions as measures is more challenging than other subconstructs, especially when it involves using number lines (Charalambous & Pitta-Pantazi, 2007; Izsák et al., 2008).

Partitioning skills can also be developed by unitizing (Lamon, 1996, 2012). Unitizing is a cognitive process for conceptualizing a given quantity as a unit or whole (McCloskey & Norton, 2009; Norton & McCloskey, 2008). For example, setting two isosceles triangles into a single unit using a patterned block. These activities can run simultaneously in developing the concept of fractions, especially part-whole relationships (Pantziara & Philippou, 2012), equivalent fractions, and in turn, developing the concept of fractions as quotients (Lamon, 2012). There are several activities in developing unitizing, one of which asks students to generate equivalent expressions for the same quantity. For example, there are 24 beads, and it will be equivalent to two groups of 12 beads or three groups of 8 beads. It can also be developed with fractions as operators because students will be asked to determine the unit before operating it.

Partitioning, iterating, and unitizing activities play an essential role in developing students' multiplicative reasoning (Singh, 2000) and proportional reasoning (Lamon, 1996; Langrall & Swafford, 2000; Purnomo, 2015). Besides, when students learn fractions as ratios, proportional reasoning might be increasingly apparent. The ratio is the core of proportional reasoning and the problem of proportionality (Purnomo, 2015). Langrall and Swafford (2000) revealed that there are at least four essential prerequisites in proportional reasoning, including: (1) developing increasingly complex unitizing, (2) recognizing situations and reasonable or appropriate ratio, (3) understanding two different forms of the ratio which might not necessarily have different values, and (4) recognizing the difference between additive (relative) and multiplicative (absolute) relationships. Furthermore, Purnomo (2015) states that the development of proportional reasoning can be carried out initially by constructing a foundation for understanding fractions, decimals, percentages, ratios and connecting them and related contexts.

Present Study

We see that the relationship among subconstructs is complex and dynamic, so which is the best possible subconstruct to introduce fractions? Other researchers also asked this question (Lamon, 2007; Watanabe et al., 2017), and the answer to it requires an empirical study. Other possible questions are: Is it better for students to be exposed to all five subconstructs early, or is it better to focus on one (beyond part-whole)? If it is better to focus on one, which? Do students need to understand all five subconstructs before Algebra? Traditionally, fractions as part-whole were the first to be introduced in most countries' curricula and textbooks (Rahmawati et al., 2020). However, some researchers criticize it (Simon et al., 2018; Watanabe, 2006; Wilkins & Norton, 2018), and in fact, the curriculum is more focused and only limited to part-whole subconstruct (Rahmawati et al., 2020). Simon et al. (2018) revealed that fractions as measures are more effective than part-whole and provides several benefits. However, studies also explain that students encounter obstacles using number lines (Charalambous & Pitta-Pantazi, 2007; Izsák et al., 2008).

Besides, we agree with the constructivist view that students will construct knowledge easily when exposed to related prior knowledge and experience. Therefore, we started with fractions as a quotient. We use the child's experience by considering the whole number's division to develop the partition concept. Furthermore, Hackenberg (2010) also states that fractions as quotients have not received more attention in fractions research. In other words, the research literature has not been found that considers division as the starting point for learning fractions.

Similar to previous studies (Sari et al., 2020; Simon et al., 2018; Steffe, 2004), in this article, we solely focus on the teaching experiment findings associated with the revised HLT. In the first HLT, we use fractions as quotients as a starting point, then it is followed by fractions as part-whole, fractions as ratios, fractions as measures, and it ends with fractions as operators. However, in the teaching experiment session, we revised this HLT by placing the fraction as ratios at the end by referring to the findings of the preliminary teaching experiment that this section involves the complexity of other subconstructs, in addition to the cognitive characteristics of students who are not ready for this material and the goal of establishing proportional reasoning. The revised HLT can be visualized in Figure 2.

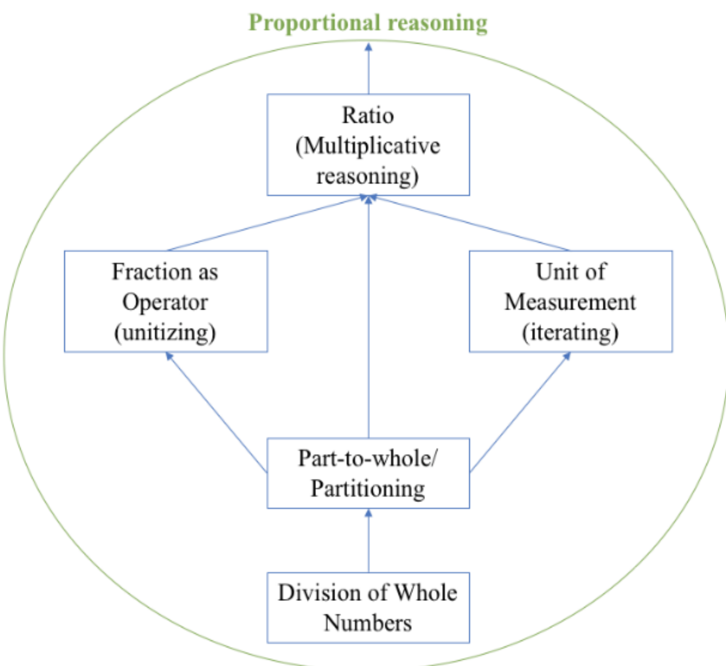

Figure 2. The Fractions Subconstructs Learning trajectory

Figure 2 illustrates the path through which the research objectives are reached. Each learning session has its sub-goals and supports the primary goal. The objectives of each learning session are: (1) students can develop concepts of fair sharing using their experiences; (2) students can interpret the fair share of a whole through partitioning activities; (3) students can use fractions as a unit of measurement and practice their skills in iterating; (4) students can apply the concept of fractions as a function of a quantity and develop unitizing skills; (5) students can develop multiplicative reasoning. Objectives (1) and (2) relate to how to introduce fractions and construct the concept of partitioning by linking their experiences when learning division, then the partitioning concept is adopted to be applied for purposes (3) and (4). Then each of the sub-objectives is simultaneously used to apply the ratio concept (5). As such, a series of sub-goals directs to promote proportional reasoning.

METHOD

Design research was applied to reach the objective of the study. It includes three phases: (1) preliminary teaching experiment, (2) teaching experiment, and (3) retrospective analysis (Gravemeijer, 2004; Gravemeijer & Cobb, 2006). However, to focus on the research objectives effectively and efficiently, this article focuses on reporting the revised HLT.

Participant

The participants in this study were 30 fourth-grade students at one public elementary school in Jakarta City. These 30 students were involved in the teaching experiment stage. They consisted of 17 girls and 13 boys with an average age range of 9 - 10.

The participants learned straightforward fractions (e.g., 1/2, 1/3, ¼) at the second grade. In the third grade, they continued developing that concept and then learned the addition and subtraction of fractions in the fourth grade. Nevertheless, according to their teacher, the participants have a weak understanding of the concept of fractions. It may hamper them from dealing with applying the concept of fractional operations in the subsequent grades.

Intervention

Researchers conducted a preliminary teaching experiment based on a hypothetical learning trajectory (HLT) in the first cycle. Based on the first cycle in five teaching sessions, the HLT 2 was generated as the refinement of the first HLT. HLT 2 was then implemented in the teaching experiment classroom and used as a guide for teaching practices.

The teaching experiment was conducted in a real classroom that consists of thirty 4th grade students. The conducted nine-stages learning activities are pretest, interview after pretest, learning

1 (fractions as quotients), learning 2 (fractions as part-whole), learning 3 (fractions as measures), learning 4 (fractions as operators), learning 5 (fractions as ratios), posttest, and interview after posttest.

Data Collection and Analysis

First, this phase formulates conjectures about what happened and examines it using the available data. Second, this phase formulates conjectures as to why this happened and then examines it. In formulating and testing conjectures, all collected data is considered, especially the transcription of whole-class discussion and students' work. The retrospective analysis phase was carried out based on all collected data during the teaching experiment. In this phase, researchers employ HLT as a guide in answering research questions. Subsequently, researchers develop local instructional theories and apply them to more general research topics.

The data was collected through tests, interviews, observations, and documentation. Before and after the intervention, we conducted tests and interviews to determine the improvement of HLT. In this study, an unstructured interview was carried out, and during observation, the role of researchers is as a participant-observer. All events that occurred were documented and recorded by two cameras. The first camera is static (static camera), which is intended to record all activities in the classroom, and the rest is dynamic (dynamic camera) to record particular activities during classroom discussions—professional videographers involved in this process.

Concerning internal validity in this study, the retrospective analysis is applied by researchers to interpret data obtained from interviews, observations, and documentation. The instrument used was examined before the implementation phase, then analysis was carried out to test the assumptions about students' thinking and learning processes. Besides, important events in each part of the learning process were recorded through video recording to obtain meaningful contexts. Furthermore, external validity focuses on the results obtained in different situations, guided by the question of how particular elements of the results obtained will apply to other situations. In this study, researchers try to improve reliability by discussing necessary findings in design experiments among researchers and improving the findings simultaneously.

RESULTS AND DISCUSSION

Teaching experiment 1: Transitioning from the whole number to fractions by student experiences

We begin this trajectory by introducing the meaning of fractions as division. We refer to the results of previous studies which show that students are weak in the concept of "equal share" (Purnomo et al., 2014, 2019) in which it causes learning obstacles to the application of other fraction concepts, such as sorting and comparing fractions and also fraction operations. It is pretty challenging because, in general, the part-whole approach is first used to introduce the concept of fractions. Moreover, in Indonesia, elementary school mathematics textbooks emphasize this part-whole approach to introducing the concept of fractions rather than other subconstructs (Purnomo, 2015; Rahmawati et al., 2020; Wijaya, 2017). Nonetheless, we consider our students' experiences when learning fractions before the part-whole concept using the whole number division as a foundation. Besides, a smooth transition between numbers and fractions becomes critical to develop proportional reasoning abilities (Im & Jitendra, 2020).

At learning 1, the teacher employs the context of the division of three crackers to six people through pieces of paper provided. Different strategies are found based on student group discussions in determining pieces in which they indicate different pieces and equal sizes. Some students divide the first crackers into six parts, followed by the second and third crackers, and then distribute them so that each person gets three-one-sixth. Meanwhile, some of them divide three crackers into six people by splitting each cracker into two parts so that each person gets half. The teacher then asks one group to show that division is to produce equal pieces (see Figure 3).

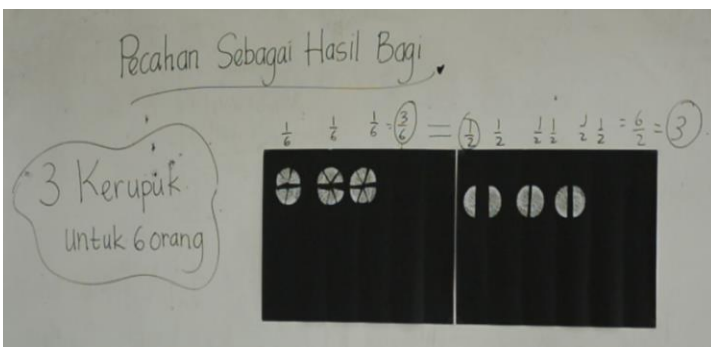

Figure 3. An example of group work in showing two different strategies for distributing crackers

Figure 3 shows that students use two strategies, in which the left part indicates distributive partitioning, while the right one indicates non-formal partitioning (halving). Empson (see in Jones, 2012; Purnomo, 2015) stated that children already know halves and do problems involving repeated halving without formal instruction. This partitioning activity simultaneously might develop students' iterating ability and the sense of equivalent fractions.

Furthermore, students are engaged with complex problems and guided with three worksheets adapted activities suggested by Lamon (2012). The first worksheet consists of four problems that ask students to work individually and carry out a fair sharing of a certain quantity to a number of people by sticking and or illustrating the pieces in the answer column. The activity aims to develop partitioning skills. The second worksheet includes five questions that are discussed in groups. It requires students to work together in determining the way how to share by sketching the pictures. Finally, the third worksheet includes three descriptive questions that must be answered individually by students. Students are asked to think of alternatives in dividing cakes into unequal but congruent forms on this worksheet. Figure 4 is a sample of student work on the first worksheet.

Figure 4. A sample of children's work for the first worksheet

Translaione:

Task 1: What *are the steps to share five cakes fairly with four friends? Explain with illustrations!*

Figure 4 illustrates the preserved-pieces partitioning strategy (Lamon, 1996), which shows students' strategy in distributing five pieces of bread by distributing them one by one to get one piece of bread and then dividing one remaining bread into four people. Hence, students get one piece and ¼ pieces. Furthermore, the activity on this worksheet guides students to partition according to their experience so that their answers vary. An idea appears to divide each bread into four at once and then distributed—each person gets five a quarter—known as distributive partitioning. Some are cutting it half by half, often referred to as halving—each person gets two half one quarter. Thus, in addition to developing partitioning skills, this activity helps lay the foundation for learning equivalent fractions, addition, and division of fractions in the future.

Teaching experiment 2: Partitioning activity

In learning 2, the teacher delivers students to learn the meaning of fractions as part-whole. The teacher starts by giving a part-whole case using a hidden grid flat shape (the piece looks unequal), as shown in Fig. 5. This problem is given since they have learned the concept of simple fractions in the previous class. It is essential to anticipate the child's variety of thinking about fractions as part-whole and develop children's thinking habits and sensitivity to fractions.

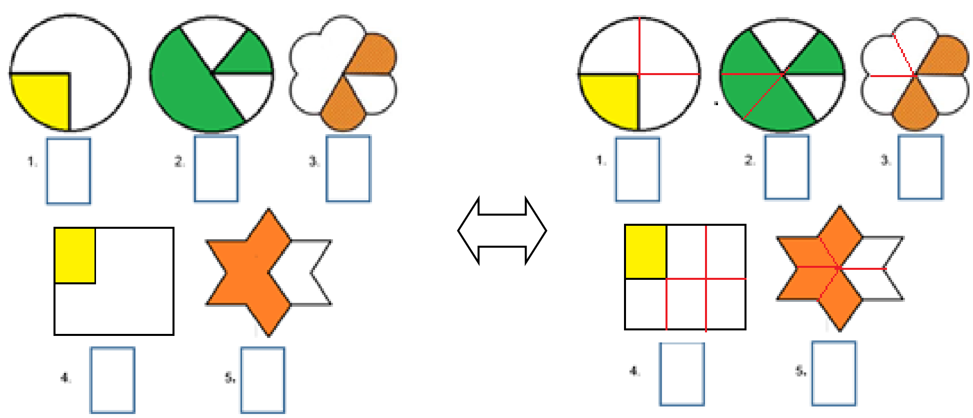

Figure 5. The problem of the fractional part is not the same as the invisible line

R: What is the fractional value in figure number 1, S1?

S1: One-fourth

R: Then who knows what fraction is this (no. 2)? (All students are silent)

What if this one (no. 3) (all students are still silent)? Is it hard?

R: Now pay attention! If I draw lines, can you read the fractions after this? Try No. 2. What is the fractional value?

S2: four sixth

R: ok, the third one?

S3: two sixths

R: correct, last numbers 4 and 5

S4: one-sixth

S5: four-sixth

R: It becomes easy. Why?

S4 & S5: Because there is a hint line

Most students have difficulty solving the problem as the image of fractions does not match their previous mental image, in which the part and the whole structure are regular. For this reason, the teacher provides a stimulus by asking students to match the group of shapes next to them and asking students to conclude. Through the question and answer process and adding helplines that divide the flat shape of the same size, students can determine the fractional value for the colored part. This activity can certainly be used when encountering the same problem, namely marking it with a hint line or their mental image.

The next activity was carried out by developing their understanding of the fractional part's concept with a hidden grid. There are four worksheets to accommodate these activities. The first worksheet includes one question that asks students individually to take a propositional attitude and explain their reasons for the six shaded parts of the structure. The second worksheet involves arranging and matching the available shapes (i.e., equilateral triangles, rhombus, parallelogram, trapezoid). It is carried out to form the required fractions of a hat-shaped shape and place it on a paper-lined with triangular grids. This problem adopts the activities suggested by Roddick and Silvas-Centeno (2007). The third worksheet consists of three description questions that are relevant to their daily experiences. The fourth worksheet contains five-word problems that students must answer individually in with each item requires students to use reasoning and estimation. The following is sample student work in learning 2.

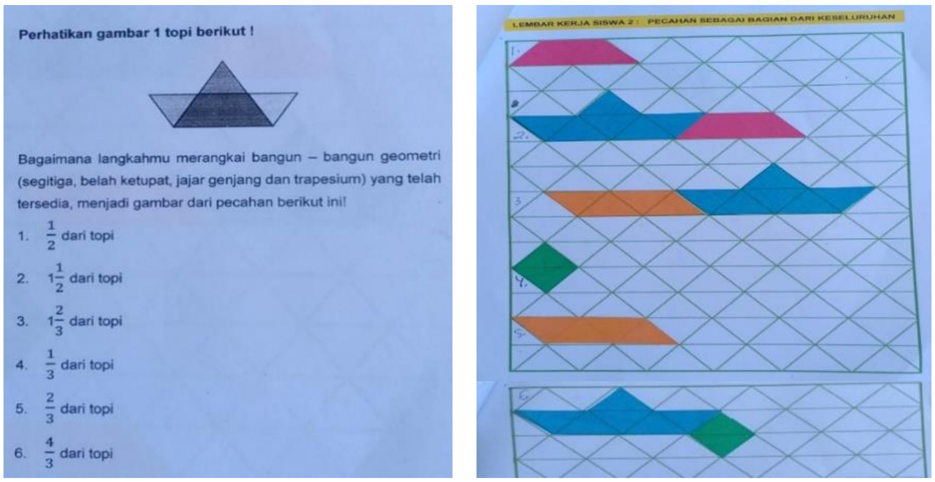

Figure 6. A sample of group work in stringing and matching activities (block patterned)

Translation:

How do you arrange the geometric shapes given (triangles, rhombus, parallelogram, and trapezoid) to become a figure of the following fractions: 1/2, $1\frac{1}{2}$, $1\frac{2}{3}$, 1/3, 2/3, *and* 4/3 *of the hat?*

Figure 6 shows that students use patterned blocks to determine the correct piece and whole. When they are asked to show 1/2 of the hat, they have to find the whole shape of the chosen shape. When the triangle is chosen as a unit, they find six triangles in the hat, so they will pair three of them to represent 1/2. There is also a pair of one trapezoid isosceles for 1/2 because one trapezoid is equivalent to 3 triangles. Various student representations were obtained when completing this activity.

In this activity, students are capable of determining the whole of the part from the fractional part and developing their partitioning skills to determine the fractional part of the whole of the part. In other words, students focus on the part of the fractions and the relationship to the whole. This ability can be called *unitizing* (Lamon, 1996). Unitizing is the mental ability to assign units different from the size of a given quantity (Lamon, 1996, 2012; Purnomo, 2015). This activity also bridges them to learn equivalent fractions (Purnomo, 2015).

Teaching experiment 3: Iterating activity

We introduce the concept of fractions as measures to develop students' sensitivity regarding fractions as a quantity (Watanabe, 2006), developing proficiency in additive operations on fractions (Charalambous & Pitta-Pantazi, 2007), and developing their partitioning and iterating skills that are useful for multiplicative reasoning (Singh, 2000). Therefore, this concept is often introduced with the use of number lines or other measurement devices.

In the beginning, students are given a number line, and the teacher asks students to partition until they can mention the fractional value at the point marked on the number line. Under previous predictions, that students will have difficulty in partitioning gave number lines. This is supported by several researchers (Charalambous & Pitta-Pantazi, 2007; Purnomo et al., 2014) who found that students encounter difficulty when working with the number lines such as determining the location or position of fractions, calculating partition marks rather than intervals, determining the fraction unit, and determining the fraction between two fractions in a number line. Therefore, we ask students to use a ruler and partition it into four sections following the mark on the ruler. This method is enough to help students understand how many fractions are requested (i.e., 1/4).

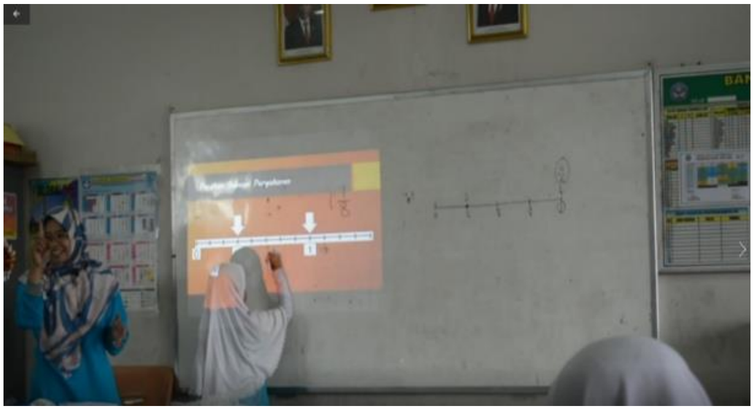

Figure 7. Students use iteration activities

To understand how to name the fraction value on the number line, the teacher gives a probing question by asking students to partition it into a different number of 8 and asking how many fractions were marked beforehand. Figure 7 shows that students determine 3/8 and from a number line. Students do iterate that there are 8 of 1/8s units in 1 unit and sort the positions of 1/8, 2/8, 3/8, and $1\frac{1}{8}$ simultaneously.

We further explore the concept of fractions as measures using two design tasks. The first task requires students to measure a specific fraction starting from the fraction unit using origami paper media. Activities on this task are intended to use various media, refine paper folding activities, and use number lines in the first cycle. Next, the second task requires students to determine a known fraction on the number line and wants students to determine the fraction between two numbers. Here is one sample of student answers when working on the first task.

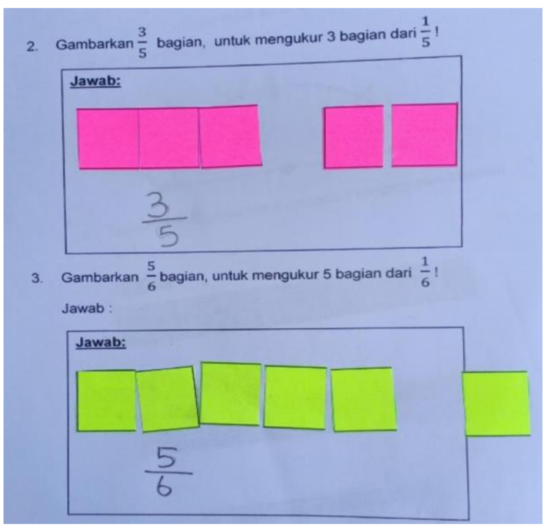

Figure 8. A sample of answers from measuring activities

Figure 8 shows activities that involve partitioning and iterating skills (measurement). For example, in case number 2, students prepare five origami papers (partitioning) to measure 3 of 1/5s to get 3/5 (iterating) and case number 3, dividing six origami papers and counting 5 out of 1/6s to get 5/6. This activity becomes essential because it requires children to focus on partitioning skills requiring them to focus on parts (intervals) of origami paper rather than partition markings such as number lines. This activity is helpful to avoid students' misconceptions about measurement(c.f. Wijaya, 2008). This activity also involves the iterating activity, copying the fractional unit to obtain the larger portion requested. The partitioning and iterating activities involved in learning fractions as measures become important in mathematics classrooms. It helps students develop the concept of fractions and develop multiplicative reasoning skills, which are the basis of proportional reasoning.

Teaching experiment 4: Unitizing activity

In learning 4, the teacher uses the button to introduce the concept of fractions as operators. Fractions as operators have been implemented by students in their daily lives even though they have not yet learned fractions formally. Students are familiar with the sentence: ask for half? One quarter? Or a fraction of this? These informal questions are adopted to raise context-based issues relating to fractions as operators. The teacher asks students to determine a fraction of the 24 buttons that each student has brought, which is as follows.

1. What is half of the 24 buttons?

2. What is one-third of the 24 buttons?

3. What is a quarter of 24 buttons?

Figure 9. Activity to group 24 buttons based on specific fractions

Figure 9 shows how a child groups 24 buttons into specific fractions according to the teacher's instructions. When the instructions ask them to look for 1/2 of the 24 buttons, with the teacher's direction, they group them into two equal parts so that 1 of 2 groups of buttons is 12. Similarly, when the instruction requires 1/4 out of 24, they grouped them into four equal parts. With this activity, children can use their partitioning and unitizing skills. Partitioning skills occur when requiring students to make the same number of parts as a whole, while unitizing is seen when students do 24 group buttons into 2 (12 buttons), 3 (8 buttons), and 4 (6 buttons).

Furthermore, the teacher also develops students' sensitivity to fractions as operators using number machines like Figure 10. This activity adopts the idea of Lamon (2012).

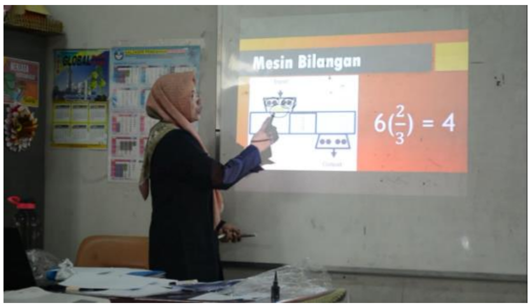

Figure 10. Machine numbers for fractions as operators

In Figure 10, the teacher demonstrates a number machine to determine the operator that converts input 8 into output 4. The transformation process of input and output is carried out as the previous one. However, this machine draws students closer to the function of fractions as operators, namely transforming from a certain quantity (input) and produce another quantity (output).

Teaching experiment 5: Transitioning additive to multiplicative reasoning

Before understanding fractions as ratios, the teacher introduces multiplicative relationships using the definition of ratio. The ratio is a multiplicative comparison of two quantities or sizes, written with $a : b$ (Purnomo, 2015). Multiplicative reasoning requires students to differentiate between multiplicative relationships (multiplication/division) and additive relationships (addition/subtraction) between two quantities. This context is then used to ask students to identify the relationship between nine green and three orange balls, as illustrated in Figure 11.

Figure 11. The teacher makes open questions related to the relationship between the number of orange and green balls

Figure 11 shows one activity identifying the relationship between the number of orange and green balls. An example of student answers is "*the number of green balls is more than the number of orange balls, the number of green and orange balls is 12, the number of green balls is six more balls than orange balls, the number of green balls is three times the number of orange balls, the number of balls orange quarter of the whole ball*". These responses can be expressed $x > y$; $x + y = 12$; $x = y + 6$; $x = 3 \times y$; $x = \frac{1}{4}z$ respectively, so that it can be used to differentiate between additive or multiplicative relationships. This mathematical statement which includes multiplication and division relations (known as multiplicative relations, i.e. $x = 3 \times y$ and $x = \frac{1}{4}z$) is what is then introduced as a ratio. It is essential to bridge the transition of additive to multiplicative reasoning. Through this activity, students also learn about some variations in ratios that are appropriate and non-ratios as one of the essential components in proportional reasoning (Lamon, 2012; Langrall & Swafford, 2000; Purnomo, 2015). Furthermore, the teacher provides reinforcement that the ratios stated previously have differences. The first case is the type of ratio as a part-whole relationship — *the number of orange balls a quarter of all balls or* $(x = \frac{1}{4}z)$ *or* $1 : 4$ — which is also known as a fraction as previously learned. The second case is the type of ratio as part relationships — *the number of green balls three times the number of orange balls or* $x = 3 \times y$ *or* $3{:}1$ — which is the ratio but not the fraction. The same activity was continued with varying numbers and colors of balls to develop their sensitivity to the meaning of fractions as ratios.

Armed with their understanding of value fractions, order fractions, and fraction comparisons, we began to introduce the concepts of propositions and non-propositions, namely by presenting pairs of fractions that were equivalent to 3/4 and 5/6. Students were asked to see whether they were the same or different. However, all students were silent and could not answer the questions because they were confused. The teacher gives a demonstration using the media bar on the board and compares it.

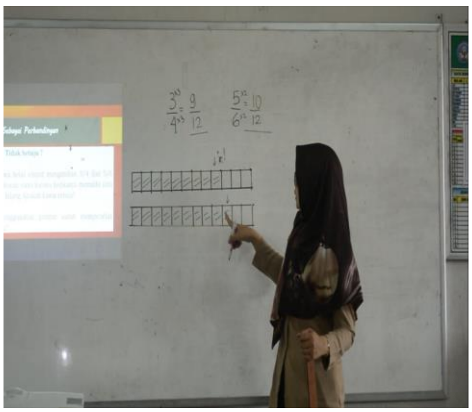

Figure 12. The teacher uses the media bar to do iterating

Figure 12 shows that the teacher performs iterating bars of multiples of four and six to get the same unity. The top bar is obtained by iterating three times, while the bottom bar is twice. After getting the same unity, the teacher compares 3/4 and 5/6 by pointing to 3 copies of the 3/4 top bar and two copies of the 5/6 bottom bar. With shading, that comparison is equivalent to 9/12 and 10/12. This activity indicates that partitioning, iterating, and unitizing are helpful in completing fraction comparisons, developing their multiplicative reasoning, and establishing their proportional reasoning.

CONCLUSIONS

Design research is carried out to provide a general overview for teachers about instructional sequences in student learning and learning processes. This study's findings indicate that the developed learning trajectory can lay essential foundations in assisting fractions concept development. There are five main learning activities as a trajectory to promote it, namely (1) starting with introducing fractions as the division of whole numbers; in fair sharing, the obtained pieces do not have to be congruent; (2) exploring and developing partitioning activities on the concept of fractions as part-whole; (3) using the concept of partitioning to get a relative unit of measure from fractions and to transition from discrete concepts to continuous concepts; consider fractions as a systematic measurement related to the use of a number line or other measurement device; (4) employing partitioning capabilities to unitize and operate numbers; (5) using partitioning, unitizing, and carrying out operations as the basis for multiplicative relationships and proportional reasoning.

There are several key features in our research findings in laying the foundation for developing the concept of fractions, namely (1) the importance of the transition between the whole numbers and fractions so that using the prior experience is the best alternative. In the context of our study, namely by using whole number division to introduce the concept of fractions; (2) the importance of the concept of partitioning for the five subconstructs, for example, the students learned the quotient sub-construct while finding the fair share of 3 crackers among six people. They accomplished this by an informal halving technique (informal operator) while others divide each successive cracker into six parts and then distribute. Thus, this second reasoning process involved part-whole partitioning followed by coordination – distribution and was thus as a learning method was a good application of part-whole partitioning within the quotient subconstruct; (3) the importance of bridging the transition of additive and multiplicative reasoning, so we use the ways that students identify these two traits with independent and differentiating investigations and use them to conceptualize the concept of ratios.

Following the research design's objectives, the implications of this study should be considered by teachers. Teachers are expected to go out of their routines and explore the concept of fractional parts that are not focused only on fractions as part-whole relationships, even when introducing it to students for the first time. This study suggests starting with fractions as division, which they have learned before learning fractions. We focus on this, even though other subconstructs are also very attached to their daily lives. We also find that the findings of this research help prepare a mathematics curriculum based on research-based learning trajectories rather than merely adapting or copying curricula in other advanced places. It is important because each student and teacher must be exposed to teaching and learning situation which fit their characteristics and their respective environments.

Our study's limitation is the time constraint in which we conducted only five preliminary teaching and five teaching experiments. It has implications for the development of each subconstructs, which needs to explore more free play activities. Therefore, other researchers need to manage this and discuss it with responsible parties. We also delimit the critical aspects of proportional reasoning developed through subconstructs of fractions so that other researchers can use this research to enrich cases directly related to the problem of proportionality or others. We also suggest a need for substantial research to determine the order of mathematical content for each grade level. As it might be a complicated effort, thus researchers and teachers need to work collaboratively and guarantee research-based practice based on students' context.

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REFERENCES

[1]Behr, M. J., Lesh, R., Post, T., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–126). Academic Press Inc.

[2] Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a Theoretical Model to Study Students' Understandings of Fractions. *Educational Studies in Mathematics, 64*(3), 293–316. https://doi.org/10.1007/s10649-006-9036-2

[3] Clarke, D., Roche, A., & Mitchell, A. (2011). One-To-One Student Interviews Provide Powerful Insights and Clear Focus for the Teaching of Fractions in the Middle Years. In J. Way & J. Bobis (Eds.), *Fractions: Teaching for Understanding* (pp. 23–41). Australian Association of Mathematics Teachers.

[4] Clements, D. H., Sarama, J., Baroody, A. J., & Joswick, C. (2020). Efficacy of a learning trajectory approach compared to a teach-to-target approach for addition and subtraction. *ZDM*. https://doi.org/10.1007/s11858-019-01122-z

[5] Clements, D. H., Sarama, J., Baroody, A. J., Joswick, C., & Wolfe, C. B. (2019). Evaluating the Efficacy of a Learning Trajectory for Early Shape Composition. *American Educational Research Journal*, *56*(6), 2509–2530. https://doi.org/10.3102/0002831219842788

[6] Gravemeijer, K. (2004). Local Instruction Theories as Means of Support for Teachers in Reform Mathematics Education. *Mathematical Thinking and Learning*, *6*(2), 105–128. https://doi.org/10.1207/s15327833mtl0602_3

[7] Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In *Educational design research*. Routledge.

[8] Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction*, *28*(4), 383–432. https://doi.org/10.1080/07370008.2010.511565

[9] Im, S., & Jitendra, A. K. (2020). Analysis of proportional reasoning and misconceptions among students with mathematical learning disabilities. *The Journal of Mathematical Behavior*, *57*, 100753. https://doi.org/10.1016/j.jmathb.2019.100753

[10] Izsák, A., Tillema, E., & Tunç-Pekkan, Z. (2008). Teaching and learning fraction addition on number lines. *Journal for Research in Mathematics Education*, *39*(1), 33–62. https://doi.org/10.2307/30034887

[11] Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101–144).

[12] Lamon, S. J. (1996). The Development of Unitizing: Its Role in Children's Partitioning Strategies. *Journal for Research in Mathematics Education*, *27*(2), 170–193. https://doi.org/10.2307/749599

[13] Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In J. Frank K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 1, pp. 629–667). Information Age Publishing Inc.

[14] Lamon, S. J. (2012). Teaching Fractions and Ratios for Understanding. In *Teaching Fractions and Ratios for Understanding* (3rd Ed.). Routledge.

[15] Langrall, C. W., & Swafford, J. (2000). Three Balloons for Two Dollars: Developing Proportional Reasoning. *Mathematics Teaching in the Middle School*, *6*(4), 254–261. www.nctm.org.

[16] McCloskey, A. V, & Norton, A. H. (2009). Using Steffe Advance Fractions Scheme.

Mathematics Teaching in the Middle School, *15*(2).

[17] Norton, A. H., & McCloskey, A. V. (2008). Modeling students' mathematics using Steffe's fraction schemes. *Teaching Children Mathematics*, *15*(1), 48–54.

[18] Pantziara, M., & Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics*, *79*(1), 61–83. https://doi.org/10.1007/s10649-011-9338-x

[19] Purnomo, Y. W. (2015). *Pembelajaran Matematika untuk PGSD: Bagaimana Guru Mengembangkan Penalaran Proporsional Siswa*. Erlangga.

[20] Purnomo, Y. W., Kowiyah, Alyani, F., & Assiti, S. S. (2014). Assessing number sense performance of Indonesian elementary school students. *International Education Studies*, *7*(8), 74–84. https://doi.org/10.5539/ies.v7n8p74

[21] Purnomo, Y. W., Widowati, C., Aziz, T. A., & Pramudiani, P. (2017). Fractions division knowledge of elementary school student: The case of Lala. *AIP Conference Proceedings*, *1868*. https://doi.org/10.1063/1.4995148

[22] Purnomo, Y. W., Widowati, C., & Ulfah, S. (2019). Incomprehension of the Indonesian Elementary School Students on Fraction Division Problem. *Infinity Journal*, *8*(1), 57–74. https://doi.org/10.22460/infinity.v8i1.p57-74

[23] Rahmawati, T., Pangesti, S. R., Nuriadin, I., Kurniasih, M. D., & Purnomo, Y. W. (2020). How do Indonesian elementary school mathematics textbooks introduce fractions? *Journal of Physics: Conference Series*, *1581*, 012024. https://doi.org/10.1088/1742-6596/1581/1/012024

[24] Roddick, C., & Silvas-Centeno, C. (2007). Developing Understanding of Fractions through Pattern: Blocks and Fair Trade. *Teaching Children Mathematics*, *14*(3), 140–145.

[25] Sari, P., Hajizah, M. N., & Purwanto, S. (2020). The neutralization on an empty number line model for integer additions and subtractions: Is it helpful? *Journal on Mathematics Education*, *11*(1), 1–16. https://doi.org/10.22342/jme.11.1.9781.1-16

[26] Shin, J., & Lee, S. J. (2018). The alignment of student fraction learning with textbooks in Korea and the United States. *The Journal of Mathematical Behavior*, *51*, 129–149. https://doi.org/10.1016/j.jmathb.2017.11.005

[27] Siebert, D., & Gaskin, N. (2006). Creating, Naming, and Justifying Fractions. *Teaching Children Mathematics*, *12*(8), 394–400.

[28] Simon, M. A., Placa, N., Avitzur, A., & Kara, M. (2018). Promoting a concept of fraction-as-measure: A study of the Learning Through Activity research program. *Journal of Mathematical Behavior*, *52*, 122–133. https://doi.org/10.1016/j.jmathb.2018.03.004

[29] Singh, P. (2000). Understanding the concepts of proportion and ratio constructed by two grade six students. *Educational Studies in Mathematics*, *43*(3), 271–292. https://doi.org/10.1023/A:1011976904850

[30] Steffe, L. P. (2004). On the Construction of Learning Trajectories of Children: The Case of Commensurate Fractions. *Mathematical Thinking and Learning*, *6*(2), 129–162. https://doi.org/10.1207/s15327833mtl0602_4

[31] Steffe, L. P., & Olive, J. (2010). *Children's Fractional Knowledge*. Springer US. https://doi.org/10.1007/978-1-4419-0591-8

[32] Tsai, T. L., & Li, H. C. (2017). Towards a framework for developing students' fraction proficiency. *International Journal of Mathematical Education in Science and Technology*, *48*(2), 244–255. https://doi.org/10.1080/0020739X.2016.1238520

[33] Watanabe, T. (2006). The teaching and learning of fractions A Japanese perspective. *Teaching Children Mathematics*, *12*(7), 368–374.

[34] Watanabe, T., Lo, J.-J., & Son, J.-W. (2017). Intended Treatment of Fractions and Fraction Operations in Mathematics Curricula from Japan, Korea, and Taiwan. In J.-W. Son, T. Watanabe, & J.-J. Lo (Eds.), *What Matters? Research Trends in International Comparative Studies in Mathematics Education* (pp. 33–61). Springer International Publishing. https://doi.org/10.1007/978-3-319-51187-0_2

[35] Wijaya, A. (2008). *Design Research in Mathematics Education: Indonesian Traditional Games as Means to Support Second Graders Learning of Linear Measurement*. Universiteit Utrecht.

[36] Wijaya, A. (2017). The Relationships between Indonesian Fourth Graders' Difficulties in Fractions and the Opportunity to Learn Fractions: A Snapshot of TIMSS Results. *International Journal of Instruction*, *10*(4), 221–236.

[37] Wilkins, J. L. M., & Norton, A. (2018). Learning progression toward a measurement concept of fractions. *International Journal of STEM Education*, *5*(1). https://doi.org/10.1186/s40594-018-0119-2