Editorial from Mónica Arnal Palacián, Didactics Editor of MTRJ

The expansion of the Mathematics Teaching Research Journal in recent months is reflected not only in the origin of the authors of the manuscripts in this issue, but also in the different themes and methodologies presented and closely connected to mathematics education today. The reader will be able to find research on geometric and calculus notions, studies emphasizing the concerns of the mathematics teacher in the classroom, analysis of textbooks and mathematical challenges, among others. It is also important to note that the studies presented present both quantitative and qualitative methodologies.

The difficulties faced by education in different parts of the world as a result of covid-19 and, consequently, the different face-to-face and virtual situations in the teaching-learning processes, have not prevented research in mathematics education from continuing to develop with great quality. Proof of this is the research developed in the ten manuscripts that make up this issue, which are described below.

The first manuscript in this issue “Exploring the Relationship among Mathematics Attitude, Gender, and Achievement of Undergraduate Health Science Students” is presented by Ayebo and Dingel from the University of Minnesota Rochester, USA. They bring together two current issues in mathematics education: attitudes and gender. In their article, the possible gender differences in students' attitudes towards mathematics and how these attitudes influence their performance can be found.

Secondly, Aliu, Rexhepi and Iseni, from the University "Mother Tereza"- Skopje, Macedonia, present the research entitled “Analysis and comparison of commitment, homework, extra hours, preliminary grades and testing of students in Mathematics using linear regression model”. This is a quantitative study in which they analyze how homework, commitment in the classroom, extra hours, and preliminary grades influence the student's final grade.

Next, and also carrying out a quantitative study, is the article “Mathematical competence in preschool students and its relationship with intelligence, age and cognitive functions of attention, information processing speed and reaction inhibition” by Manginas, Papageorgiou, Theodorou and Iakovaki, teachers from the University of Western Macedonia. These authors analyze the relationship between flow intelligence, age, cognitive abilities and reaction inhibition with the level of mathematical competence with pre-school pupils.
Textbooks show the progress of an education system, its relation to current scientific knowledge and in particular to mathematics. Despite the current context where digitalization is increasingly present, textbooks still occupy a central place for mathematics teachers. As a result of this interest, we find in the fourth article “Analysis of the Mathematics Function Chapter in a Malaysian Foundation Level Textbook Adopted by a Public University”, presented by Gholami, Mohd Ayub and Md Yunus, from University Putra Malaysia. They present the analysis of a function chapter to subsequently prepare teachers to identify the elements of a textbook to enhance the quality of learning.

The fifth article is entitled “Understanding Proof Practices of pre-Service Mathematics Teachers in Geometry”, presented by Manero and Arnal-Bailera from the University of Zaragoza, Spain. Taking Van Hiele's levels as a fundamental pillar of their study, they consider that proof is a concern in secondary teacher education, especially in its relation to pedagogical knowledge. In the manuscript, the authors observe three different teacher profiles whose characteristics they describe.

This issue offers the first solutions to the problem posed in the previous issue, in the recently inaugurated section “The Problem Corner” directed by Iván Retamoso, editor of MTRJ. The solutions presented follow different approaches, which may enrich and improve the mathematical knowledge of the MTRJ community. Also, a new problem is posed, aiming to be a new challenge for researchers, teachers and students.

The following article, “Teaching and Learning Processes for Prospective Mathematics Teachers: The Case of Absolute Value Equations” by Jupri and Gozali from Universitas Pendidikan Indonesia. Through a qualitative study, the authors investigate the implementation of teaching-learning processes to strengthen conceptual and procedural understanding in the use of absolute value equations, mainly with the use of GeoGebra software.

Also involving absolute value is the research of Kumari, a teacher at CUNY, USA. Entitled “Students' Difficulty with Problems Involving Absolute Value, How to Tackle this Using Number line and Box Method”, the author focuses her analysis on the number line and how the visual technique called the box method helps students connect the concept and the procedure.

The article “Learning trajectory based on fractional sub-constructs: Using fractions as quotients to introduce fractions”, written by Purnomo, Arlini, Nuriadin and Aziz, from Universitas Negeri Yogyakarta, Universitas Muhammadiyah and Universitas Negeri Jakarta, Indonesia, presents the limitations of using the fraction as a part-whole. A learning path involving the concept of fraction,
its elementary operations and the development of proportional reasoning can be found in this paper.

In the tenth manuscript, Pepkolaj and Duraj from Albanian University and the University of Shkodra "Luigj Gurakuqi", Albania, present a study entitled “How substantial and efficacious is the learning of linear algebra at undergraduate level?” The authors measured the state of learning of linear algebra with engineering students after one academic year and found that the learning of linear algebra was not effective. They also found that test scores are a determinant of temporal learning.

Finally, this issue closes with “The Calculus Concept Inventory Applied to the Case of Large Groups of Differential Calculus in the Context of the Program 'Ser Pilo Paga' in Colombia” by Villalobos-Camargo from the University of Bogotá Jorge Tadeo Lozano, Colombia, contrasts the understanding of calculus ideas and the ability to perform calculations. He also indicates that the application of the Calculus Concept Inventory (CCI) improves understanding of basic concepts after taking the class.

Mónica Arnal-Palacián
Didactics Editor of MTRJ

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Exploring the Relationship among Mathematics Attitude, Gender, and Achievement of Undergraduate Health Science Students
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University of Minnesota Rochester, USA
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Abstract: The purpose of this study was to investigate the gender differences in students' attitude toward mathematics and how attitude impacts achievement in the course. The sample consisted of 172 undergraduate health science students (123 women, 49 men) enrolled in mathematics courses at a University in the Midwestern United States. Data were collected using a 20-item self-report survey adapted from the TIMSS 2011 context questionnaire. Independent sample t-test, Pearson correlation analysis and Path analysis were performed on the data. We found that there is a positive relationship between mathematics attitude and achievement. There is also a statistically significant gender difference for students liking of mathematics, with men reporting higher scores than women.

INTRODUCTION

In recent times, studies of learning mathematics have expanded to include conceptions and beliefs of mathematics (Andrews & Hatch, 2000; Cai & Wang, 2010), motivation and self-regulation (Cleary & Chen, 2009; Meyer & Turner, 2002; Schmitz & Perels, 2011), self-concept, self-esteem and self-efficacy (Bong & Skaalvik, 2003; Parker, Marsh, Ciarrochi, Marshall, & Abduljabbar, 2014; Skaalvik & Skaalvik, 2006). The general view is that people are not only cognitive individuals but also social persons with emotions and beliefs that influence their development as learners. People's behavior and choices, when confronted with a task, are
determined more by their attitudes, emotions, beliefs and personal theories, rather than by their knowledge of the specifics of the task. As such, efforts to improve mathematics education must take these attitudes into account.

The term attitude generally refers to an individual's learned tendency to respond either positively or negatively to a situation or concept, or in this case, towards mathematics (McLeod, 1994). Hart (1989) characterized attitude towards mathematics as consisting of three components: an emotional response to mathematics, a conception about mathematics, and a behavioral tendency toward mathematics. Ma and Kishor (1997) defined attitudes towards mathematics as "an aggregated measure of a liking or disliking of mathematics, a tendency to engage in or avoid mathematical activities, a belief that one is good or bad at mathematics, and a belief that mathematics is useful or useless" (p. 27).

Mathematics Attitude and Achievement

Many scholars have observed that students' attitudes are associated with their performance in mathematics. Pajares and Graham (1999) observed that mathematics self-efficacy was significantly related to the performance of middle school students. House (1993) observed that students who had higher academic self-concept earned higher grades in mathematics. Additionally, House (1995) found that several aspects of academic self-concept and achievement expectancies were significantly associated with mathematics achievement. These findings emerge across cultures and ages. In a study of high school students in Hong Kong, Rao, Moely, and Sachs (2000) observed that self-concept was a significant predictor of mathematics performance. Results of a study of elementary and middle school students showed that initial mathematics achievement was significantly related to mathematics self-concept (Skaalvik & Valas, 1999).

Though students’ mathematics achievement is linked to their attitudes toward mathematics, the direction of this relationship is not agreed upon. Some studies claim that students’ mathematics
attitude is formed and influenced by achievement in mathematics (Pajares & Graham, 1999; House, 2003). Conversely, other studies suggest that mathematics achievement occurs as a result of students’ attitude toward the subject (Di Martino & Zan, 2011; Goldin, 2002). Marsh and Young (1997) also observed that adolescent students’ academic self-concept had a significant causal effect on their mathematics achievement. Trautwein, Lüdtke, Marsh, Köller, & Baumert (2006) observed that once students gain interest in the mathematics course, they may be likely to surpass initial achievement expectations. Hence, a student's attitude can affect their achievement in mathematics. Another school of thought is that the effect one has on the other is cyclical. Abu-Hilal (2000) observed that the views of students regarding the importance of mathematics had a significant effect on their mathematics performance, and that mathematics performance results in an increase in self-concept. That is, mathematics attitude is influencing achievement and achievement is influencing mathematics attitude simultaneously. Other studies (Ma & Xu, 2004; Michelli, 2013) did not find any relationship between attitudes and achievement.

In sum, this body of research confirms the importance of attitudes on students’ performance. Further, we operate under the assumption that attitudes are not innate but result from experiences that can be changed. Students’ attitudes toward mathematics are critical to understand because they affect how invested students will be in their approach to learning mathematics, and how much enjoyment they derive from it (Moenikia, M., & Zahed-Babelan, 2010). Therefore, even moderate attitude changes could ultimately impact cognitive processing and achievement.

Mathematics Attitude and Gender

Earlier studies on mathematics attitudes reported significant gender differences in favor of men (Fennema & Sherman, 1976). However, with time, the trends became more complex and tenuous. While some studies emphasized that males showed more positive attitudes towards mathematics than females (Michelli, 2013; Tasdemir, 2009); other studies found just the opposite.
(Savas & Duru, 2005). Ma and Xu (2004) stated that both males’ and females’ attitude scores decrease in the same manner across grade levels indicating no gender difference regarding mathematics attitude among secondary school students. Schoenfeld (1989) conducted a study with high school students and found that gender differences were consistently negligible. Pajares and Graham (1999) did not find gender differences in mathematics attitudes in their study with gifted middle school students. A study conducted by Kenney-Benson, Pomerantz, Ryan and Patrick (2008) using students in 5th and 7th grades found significant differences in mathematics attitudes by gender in favor of girls. Hall (2012) found that gender gaps in mathematics attitudes and achievement at the elementary and secondary level were statistically insignificant, with boys scoring higher than girls. Research shows that gender gaps in mathematics carry on after high school. For instance, males in the United States regularly score higher on the mathematics section of the Scholastic Aptitude Test (SAT) than their female counterparts (Chubbuck et al., 2016). Also, in Turkey, males in college outperformed females in mathematics (Saygin, 2020).

A meta-analysis by Hyde et al. (1990) showed that there are gender-based inconsistencies in attitudes towards mathematics. Emotions are typically discussed in the literature on gender differences in mathematics. For example, Brush (1985) noted a feelings factor in mathematics learning, and observed that this feelings factor was highly predictive of the level of students’ course preferences.

Stipek and Gralinsky’s (1991) noted that females attributed failure to low ability, while attributing success to luck. Girls were further found to report less pride after success and a stronger desire to hide their paper after failure, a behavior interpreted as representing feelings of shame.
Aim of the Study

The aim of this study is to investigate the gender differences in students' attitudes toward mathematics and how attitudes impacts achievement in the course. The specific research questions for this study are:

1. What differences exist between the way women and men characterize their attitudes toward mathematics in a health science institution?
2. How are undergraduate health science students' attitude toward mathematics (liking, value, and confidence in mathematics) related to their mathematics achievement (measured by their final grade in the course)?

METHOD

Participants

The participants of this study were 172 undergraduate health science students (49 men, 123 women) enrolled in mathematics courses at a research university in the Midwest. The mathematics classes were College Algebra, Precalculus, and Calculus. Students were asked to complete the survey, which was provided through a link to a google form during the first week of classes. Participation was voluntary. Ethical approval for this study was granted by the University's Institutional Review Board.

Instrumentation

A 20-item survey instrument, adapted from the TIMSS 2011 contextual questionnaire (House & Telese, 2014; Khine, Al-Mutawah, & Afari, 2015) was used to measure students' attitudes toward mathematics. We used a 4-point Likert scale (1 = strongly disagree, 2 = disagree, 3 = agree, 4 = strongly agree) which consists of three 3 subscales: liking, value, and confidence. Liking measures the extent to which student report how much they like mathematics (e.g. 'I enjoy learning mathematics'). It comprises five items. Value measures students' perception of the value
they place in mathematics and its importance to their personal and professional goals (e.g. 'I need to do well in mathematics to get the job I want'). It has six items. **Confidence** measures students' perception of how confident they feel when working on mathematics problems (e.g. 'I usually do well in mathematics'). It is measured by nine items.

**Statistical analysis**

In order to answer our first research question, an independent sample *t*-test was performed to compare the mean scores of men's and women's attitude toward mathematics using IBM SPSS version 23. The second research question was addressed by means of a Pearson Correlation analysis and a Path analysis using AMOS version 23 (Arbuckle, 2015).

**RESULTS**

Table 1 summarizes means, standard deviations, and mean differences of the TIMSS 2011 constructs. The mean scores range from 2.45 to 3.61 for men, and 2.37 to 3.57 for women on a 4-point Likert scale. This shows that the sample in this study held mostly positive attitudes toward mathematics. Also, in all but three of the items (items 5, 18 and 19), men had higher scores than women.

Table 1: Mean and Standard Deviation of Survey Items (0= strongly disagree; 4 = strongly agree)

<table>
<thead>
<tr>
<th>Item</th>
<th>All (n=172)</th>
<th>Men (n=49)</th>
<th>Women (n=123)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
</tr>
<tr>
<td>Item description</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Liking</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>1. I enjoy learning mathematics.</td>
<td>3.06</td>
<td>0.7</td>
<td>3.18</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>2. I wish I did not have to study mathematics.</td>
<td>2.96</td>
<td>0.9</td>
<td>3.04</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>3. Mathematics is boring.</td>
<td>3.10</td>
<td>0.7</td>
<td>3.16</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>4. I learn many interesting things about mathematics.</td>
<td>2.94</td>
<td>0.7</td>
<td>2.98</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>5. I like mathematics.</td>
<td>2.92</td>
<td>0.8</td>
<td>2.90</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Value</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6. I think learning mathematics will help me in my daily life.</td>
<td>3.03</td>
<td>0.8</td>
<td>3.16</td>
<td>0.9</td>
<td>2.98</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>5</td>
<td>0.19</td>
</tr>
<tr>
<td>7. I need mathematics to learn other subjects.</td>
<td>3.37</td>
<td>0.7</td>
<td>3.43</td>
<td>0.7</td>
<td>3.34</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td></td>
<td>1</td>
<td></td>
<td>0</td>
<td>0.09</td>
</tr>
<tr>
<td>8. I need to do well in mathematics to get into my desired program.</td>
<td>3.53</td>
<td>0.6</td>
<td>3.65</td>
<td>0.5</td>
<td>3.48</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td></td>
<td>6</td>
<td></td>
<td>3</td>
<td>0.17</td>
</tr>
<tr>
<td>9. I need to do well in mathematics to get the job I want.</td>
<td>3.33</td>
<td>0.7</td>
<td>3.45</td>
<td>0.7</td>
<td>3.28</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td></td>
<td>9</td>
<td></td>
<td>5</td>
<td>0.16</td>
</tr>
<tr>
<td>10. I would like a job that involves using mathematics.</td>
<td>2.45</td>
<td>0.8</td>
<td>2.65</td>
<td>0.9</td>
<td>2.37</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
<td>0</td>
<td></td>
<td>2</td>
<td>0.29</td>
</tr>
</tbody>
</table>
11. It is important to do well in mathematics. | 3.61 | 0.5 | 3.71 | 0.5 | 3.57 | 0.5 | 0.15 |

Confidence

12. I usually do well in mathematics. | 3.10 | 0.7 | 3.29 | 0.7 | 3.02 | 0.7 | 0.26 |

13. Mathematics is more difficult for me than for many of my colleagues. | 2.77 | 0.9 | 2.88 | 0.9 | 2.72 | 0.9 | 0.15 |

14. Mathematics is not one of my strengths. | 2.73 | 1.0 | 2.94 | 0.9 | 2.65 | 1.0 | 0.29 |

15. I learn things quickly in mathematics. | 2.58 | 0.8 | 2.73 | 0.7 | 2.51 | 0.8 | 0.22 |

16. Mathematics makes me confused and nervous. | 2.69 | 0.9 | 2.88 | 0.8 | 2.61 | 1.0 | 0.27 |

17. I am good at working out difficult mathematics problems. | 2.47 | 0.8 | 2.73 | 0.7 | 2.37 | 0.8 | 0.37 |

18. My colleagues think I can do well in mathematics. | 2.96 | 0.7 | 2.94 | 0.7 | 2.97 | 0.6 | -0.03 |

19. My colleagues tell me I am good at mathematics. | 2.82 | 0.7 | 2.82 | 0.8 | 2.82 | 0.7 | -0.01 |

20. Mathematics is harder for me than any other subject. | 2.95 | 1.0 | 3.06 | 0.9 | 2.91 | 1.0 | 0.15 |

The results of an independent sample $t$-test showed that there were no statistically significant differences for liking, $t(170) = 0.79, p = 0.432$ and confidence, $t(170) = 1.74, p = 0.084$. Although men’s mean scores were slightly higher ($M_{\text{Liking}} = 3.55$ and $M_{\text{Confidence}} = 2.92$) than those
for women (M_{Liking} = 2.97 and M_{Confidence} = 2.73), these differences were not statistically significant (see Table 2). In contrast, a statistically significant difference was found for value, t(170) = 2.07, p < 0.05, with men scoring higher (M_{Value} = 3.34) than women (M_{Value} = 3.17). Cohen’s effect size of d = 0.34 implies a small effect.

Table 2: Mean and standard deviation of men’s and women's perception of their attitude toward mathematics and effect size estimates for differences between samples

<table>
<thead>
<tr>
<th></th>
<th>Men(49)</th>
<th>Women(123)</th>
<th>t-value</th>
<th>p-value</th>
<th>Effect Size (Cohen's d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liking</td>
<td>Mean</td>
<td>3.05</td>
<td>Mean</td>
<td>2.97</td>
<td>0.79</td>
</tr>
<tr>
<td></td>
<td>SD</td>
<td>.57</td>
<td>SD</td>
<td>.62</td>
<td>.432</td>
</tr>
<tr>
<td>Value</td>
<td>Mean</td>
<td>3.34</td>
<td>Mean</td>
<td>3.17</td>
<td>2.07</td>
</tr>
<tr>
<td></td>
<td>SD</td>
<td>.51</td>
<td>SD</td>
<td>.50</td>
<td>.040*</td>
</tr>
<tr>
<td>Confidence</td>
<td>Mean</td>
<td>2.92</td>
<td>Mean</td>
<td>2.73</td>
<td>1.74</td>
</tr>
<tr>
<td></td>
<td>SD</td>
<td>.55</td>
<td>SD</td>
<td>.66</td>
<td>.084</td>
</tr>
<tr>
<td>Overall</td>
<td>Mean</td>
<td>3.08</td>
<td>Mean</td>
<td>2.92</td>
<td>2.01</td>
</tr>
<tr>
<td></td>
<td>SD</td>
<td>.42</td>
<td>SD</td>
<td>.47</td>
<td>.046*</td>
</tr>
</tbody>
</table>

Table 3 shows the associations between students’ attitudes toward mathematics and their mathematics achievement. The correlation coefficients range from 0.02 and 0.58. There is a positive, albeit small correlation between Confidence and Grade (r = 0.257, p < 0.01). The correlations between the attitude factors of Liking and Value, and Grade are not significant.

Table 3: Pearson correlations among the affective scales and grade for all participants

<table>
<thead>
<tr>
<th>Variable</th>
<th>Grade</th>
<th>Liking</th>
<th>Value</th>
<th>Confidence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Liking  .121  
Value  .020  .367**  1  
Confidence  .257**  .583**  .223**  1  

Note: **. Correlation is significant at the 0.01 level (2-tailed).

The Path diagram is shown in Figure 1, and the fit indices for the Path analysis are shown in Table 4. The chi square value of 1.337 was not significant ($p = 0.248$), and the GFI (0.996), NFI (0.998), TLI (0.980), CFI (0.997), SRMR (0.021), and RMSEA (0.044) showed values that, taken together, suggest that the model was an excellent fit to the data.

Gender was a statistically significant predictor of both the hypothesized mediator variables of confidence (standardized regression coefficient = 0.595, unstandardized regression coefficient = 0.640 with a standard error of 0.178, $p < 0.001$) and value (standardized regression coefficient = 0.375, unstandardized regression coefficient = 0.318 with a standard error of 0.102, $p < 0.01$). Also, confidence was a statistically significant predictor of grade (standardized regression coefficient = 0.224, unstandardized regression coefficient 2.748 with a standard error of 0.929, $p < 0.01$). However, value did not influence grade in that it did not yield a statistically significant path to grade (standardized regression coefficient = -0.059, unstandardized regression coefficient = -0.912 with a standard error of 10178, $p = 0.439$).
Figure 1: Path diagram for the model

Table 4: Fit indices for the model

<table>
<thead>
<tr>
<th>Model fit indices</th>
<th>Values</th>
<th>Recommended guidelines</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^2$</td>
<td>1.337, $p = 0.248$</td>
<td>Nonsignificant</td>
</tr>
<tr>
<td>$\chi^2 / df$</td>
<td>1.337</td>
<td>$&lt; 5.00$</td>
</tr>
<tr>
<td>TLI</td>
<td>0.980</td>
<td>$\geq 0.90$</td>
</tr>
<tr>
<td>CFI</td>
<td>0.997</td>
<td>$\geq 0.90$</td>
</tr>
<tr>
<td>GFI</td>
<td>0.996</td>
<td>$\geq 0.90$</td>
</tr>
<tr>
<td>RMSEA</td>
<td>0.044</td>
<td>$\leq 0.08$</td>
</tr>
<tr>
<td>SRMR</td>
<td>0.021</td>
<td>$\leq 0.05$</td>
</tr>
</tbody>
</table>
DISCUSSION

This study investigated gender differences in students’ attitude toward mathematics as well as the effect of mathematics attitude on students' grade at an undergraduate health science institution in the Midwestern United States. The results of both the descriptive statistics and correlation analysis show that overall, students' attitudes toward mathematics had a positive relationship with their mathematics achievement. Students who reported that they usually did well in mathematics also tended to earn higher grades. Similarly, students who felt they learned things quickly in mathematics also earned higher grades. Conversely, students who reported negative comparisons of themselves to their classmates tended to earn low grades. Students who showed high achievement levels were also more likely to feel that they were good at working out difficult problems.

These findings are consistent with recent research results (Ethington, 1992; Ethington & Wolfle, 1984; Ganley & Vasilyeva, 2011; Lloyd, Walsh, & Yailagh, 2005; Nosek & Smyth, 2011). The confidence scale was found to significantly predict students’ achievement. This is consistent with the findings of Khine, Al-Mutawah, & Afari (2015). It is also important for mathematics instructors to take the necessary measures to ensure that we not only focus on the mathematics content, but also attend to the affective needs of students. In particular, teachers must ensure that they instill confidence in students who have no confidence in themselves.

We also found that there was no statistically significant difference between men and women for the mathematics attitude scales, except for liking, where men reported higher scores than women. Thus, mathematics educators need to examine practices and policies to try to understand the reason behind the existence of the gender imbalance. There may be instructional practices that unintentionally contribute to the gender difference.
Limitations and Future Studies

One limitation of this study is that a convenience sample of undergraduate students enrolled in a health science program were recruited as research participants; thus, due to the nature of the population there was a disproportionate number of females and Caucasian students in the sample. Therefore the sample was not as diverse as one would expect. Another limitation is that the study was based on self-reported data which can promote bias. Therefore, caution should be taken in generalizing this study to other samples with different demographics.

The findings from this study provide several directions for further studies. For example, additional studies are needed to determine if findings observed in this study would be apparent for students from other institutions whose settings are different from the one in this study. In addition, further research is needed to assess the relationship between students' math attitude and achievement outcomes in other academic disciplines.

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Analysis and comparison of commitment, homework, extra hours, preliminary grades and testing of students in Mathematics using linear regression model

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Abstract: In this paper, a simple and multiple linear regression model has been developed to analyze and compare math test results of two student groups in the medical high school "Rezonanca", more precisely the students of the two tenth grades in the subject of mathematics of classes X- 3 and X - 7. This paper also presents some exam exercises, which were solved by students who did not show much success in the exam, students who showed average success in the exam and those who showed excellent success. This model is based on student data, including homework, their classroom commitment, extra hours, pre-grades, and finally testing. The research in this paper shows that in reality homework activity during lessons, commitment, extra lessons and pre-test have a major impact on the student's final grade. Statistical meanings of the relationship between variables are provided. Excel and SPSS were used to obtain the results.

Keywords: Simple linear regression, multiple linear regression, homework, student commitment, extra hours, pre-grades, finally testing.

1 Introduction

Homework in Mathematics, are exercises assigned to students by their teachers, to be solved outside the classroom, respectively non-school hours. Homework is assigned for a variety of reasons such as to supplement learning activities and to practice concepts (Cooper, H., Robinson, J., Patall, E.,2006). Common homework assignments may include required reading, a written or typing project, math exercises to be completed, information to be reviewed before a knowledge testing. Some educators argue that homework is beneficial to students, as it enhances learning,
develops the skills taught in class, and lets educators verify that students comprehend their lessons (Grohnke, Kennedy, and Jake Merritt, 2016). The advantages and disadvantages of homework remain many debated issue in educational psychology. There is growing evidence that homework can be a effective addition to school learning. At the same time, it can overwhelm students, causing unpleasant emotions to both students and their parents, with negative implications for family life (Dettmers, S., Trautwein, U., Lüdtke, O., Goetz, T., Frenzel, A. C., & Pekrun, R., 2011).

In his early meta-analysis, Cooper (1989a) reported the following effect sizes (p. 71): Grades 4–6: ES = .15 (Percentile gain = 6), Grades 7–9: ES = .31 (Percentile gain = 12) and Grades 10–12: ES = .64 (Percentile gain = 24) The pattern clearly indicates that homework has smaller effects at lower grade levels and bigger effect in secondary school. Knowledge testing in addition to homework, intuitively is influenced by class commitment, extra hours and preliminary grades, and this influence was evaluated in detail using the linear regression model.

The data used in the paper was taken by the first author during the teaching hours where the preliminary test activity was evaluated, additional hours, activity in class and homework. During this observation period the following topics were studied and explained: exponentials, radicals, complex numbers, and second degree equations. The data obtained from each lesson have had a positive impact on students in terms of being more active in the classroom, because in fact the activity has also influenced the final assessment. Usually in teaching it is applied to keep an evidence for each student, and we have also applied this method. So there were a total of five exercises that had to be done by the students, the commitment in the classroom was assessed in the method that the student came up with on the board to solve the exercise or solved in his notebook. Homework were given from the textbook of Mathematics for X grade, provided according to the school curriculum and as such do not present an overload for the student. Extra hours in total have been five hours where during these hours additional exercises have been developed for students who have lagged behind but also for those who had desired to expand their mathematical knowledge. While the preliminary mark is the grade they received in the pre-test. The part of additional hours in the subject of mathematics has also been mandatory for all but especially for those students who have had stagnation in terms of units and for those students who have not shown much success during the lessons.

**1.1 Theoretical Framework**

Simple linear regression is a statistical method that represents the process of explaining the relationship between a dependent variable denoted by y and an independent variable denoted by x or the relationship between a dependent variable and more than one variable of independently with
a mathematical equation. The simple linear regression model may be suitable for many situations, but in real life to explain many models may require two or more explanatory variables (Agresti, A., 1990). Models with more than one explanatory variable are called multiple linear regression model. Multiple linear regression is used to analyze data from random comparisons, correlations, or any experimental study. Multiple linear regression is one of the most widely used statistical methods in educational research (Allison, Paul D., 1999). Multiple linear regression is defined as a multivariate technique for determining the correlation between the variable y and some of the combinations of two or more independent variables, x. The importance of these four independent variables such as homework, classroom commitment, extra hours and pre-grades actually have a very high importance and a great impact on student assessment. In fact, the student’s engagement with homework helps the student to be constantly in the course and will normally be reflected in the final test because the exercises in the test will be similar to the homework they had. According to Kohn (2006), teachers should assign homework only when they can justify that the assignments are beneficial. Kohn (2006) believes that teachers should try to involve students in deciding what homework, and how much, they should do. Homework can even give parents an opportunity to know what has been taught at school. Parents daily can see the hopeful progress or lack of progress with their child. (Costley, K. C., 2013).

Classroom commitment also has a big impact because it helps the student understand the homework in the classroom and will make it easier for them to do their homework. Also extra hours have a big impact especially for some students who may find it a little harder to accept the knowledge and learning units in mathematics, for those who catch the extra hours more easily affect the reinforcement of mathematical knowledge. The positive effects that extracurricular activities have on students are behavior, better grades, school completion, positive aspects to become successful adults, and a social aspect (Massoni, E., 2011). While pre-grades is normally a pre-test which will be similar but not the same as the final test and helps prepare students for the final test which is also crucial.

1.2 Purpose of Study
The purpose of this paper is to contribute to the knowledge about the use of simple and multiple linear regression in research in education by establishing a suitable linear regression model to analyze the relationship between the student test variable in Mathematics (where y is considered as a dependent variable) depending on homework completion, student commitment in the
classroom, extra hours and the pre-grades (where x is considered as an independent variable). The motivation of this study is to analyze the impact of homework, classroom commitment, extra hours and pre-grades on test results. Considering that many teachers do not practice this method, so continuing in the traditional teaching our motivation has been that through this analysis to present that part of these components that are mentioned in the paper are extremely important because homework in a way their constant activation in class, holding additional classes, etc., helps students to be continuously active and prepared for the test they will have at the end of the semester. Analysis for a teacher are important for getting students’ reflection on our performance. Because if they reflect positively on the homework test in overtime, etc. then it shows that the method we are practicing is also acceptable by the students but if they show lower performance then this clearly shows the strategy of achieving the lesson should be changed teaching.

Research data was collected from a sample of 36 students, 18 from class X-3 and 18 from class X-7 in the medical high school "Rezonanca" in Prishtina. In order to determine the regression coefficients and analyze the data, several mathematical software applications were used.

1.2.1 Research Methods
Our work is based on making the connection and correlation between the dependent variable which in our case is testing with each of the independent variables which are homework, classroom commitment, extra hours and preliminary grades. In order to achieve the objectives of this paper, this research question was raised with its specific hypotheses:

Hypothesis $H_0$ in the linear multiple regression model is created in the form that all regression coefficients are equal to zero ($H_0: \beta_1 = \beta_2 = \ldots = \beta_p = 0$), which means that the independent variables have no effect on the dependent variable.

Hypothesis $H_A$ is created in the form that at least one $\beta_i$ is different from zero. Which means that independent variables have an impact on the dependent variable. And in our case the $H_A$ hypothesis is expected to stand. To test statistically the significance of the parameters separately the T test is used and to test the model whether it is important as a whole the F test is used (Kallajxhë, SH, 2016; Rexhepi Sh, Iseni E, Kera S, 2021).

2. Simple and multiple linear regression model
Simple linear regression is applied to estimate the relationship between the dependent variable, y, and the single explanatory variable, x, by taking a set of data that includes observations of these two variables for a given population.
2.1 Simple linear regression model
Simple linear regression model is given by the equation:

\[ y = \beta_0 + \beta_1 x + \epsilon, \]

where

- \( \beta_0 \) presents termination on the y-axis in the population,
- \( \beta_1 \) presents population slope, and
- \( \epsilon \) presents random error.

2.2 Multiple linear regression model
The simple linear regression model may be suitable for many situations, but in real life to explain many models may require two or more explanatory variables. Models with more than one explanatory variable are called multiple linear regression models (Cochrane, D., Orcutt, GH, 1949).

Multiple linear regression model is given by the equation:

\[ y = \beta_0 + \beta_1 x_1 + \cdots + \beta_i x_i + \epsilon, \]

where

- \( y \) presents dependent variable,
- \( x_i \) presents independent variable,
- \( \beta_i \) presents estimated parameters, and
- \( \epsilon \) presents random error.

3. Data analyzed with Excel from X-7
Let us denote the independent variables such as homework, classroom commitment, extra hours and pre-grades, by \( X_1, X_2, X_3 \) and \( X_4 \) respectively, and by \( Y \) testing the dependent variable.

The equation of the multiple linear regression model is:

\[ Y = -2.18 + 2.55X_1 + 2.81X_2 + 4.81X_3 + 9.14X_4 \]

Regression statistics for the multiple regression model are presented in (Table 1). The regression equation of the simple linear model for the variable \( X_1 \) (homework) is:
Y = 3.51 + 17.14X_1

The regression statistics of the simple linear model for the variable $X_1$ is given in (Table 2).

The regression equation of the simple linear model for the variable $X_2$ (classroom commitment) is:

$Y = 8.89 + 15.88X_2$

The regression statistics of the simple linear model for the variable $X_2$ given in (Table 3).

The regression equation of the simple linear model for the variable $X_3$ (extra hours) is:

$Y = 6.65 + 18.55X_3$

The regression statistics of the simple linear model for the variable $X_3$ is given in (Table 4).

The regression equation of the simple linear model for the variable $X_4$ (pre-grades) is:

$Y = −10.84 + 20.61X_4$

The regression statistics of the simple linear model for the variable $X_4$ is given in (Table 5).

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Table 1. Multiple regression model
Here we analyze the data in Excel where the independent variables are: X1 (homework), X2 (classroom commitment), X3 (extra hours) and X4 (pre-grades) while the dependent variable is the final test result (Y). Then from the statistics obtained from the marked data we get that multiple R is 0.99 which means that the correlation between the final test result and the four independent variables is relatively high. Also R Square is 0.98 which means that only 0.02 is explained by variables which are not included in the model by random error which we explained above.

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Table 2. Simple linear regression model for variable X1 (Homework)

Here we analyze the data in Excel where homework is taken as an independent variable while as a dependent variable is the final test result. From the obtained statistical data, we see that multiple R is 0.96 which means that the relationship between the independent variable which is homework and depend variable is very close. Also, R square is 0.93 which means that 93% of the change of dependent variable is explained by the independent variable that are homework.
Table 3. Simple linear regression model for variable X2 (Classroom commitment)

For the model where the independent variable is classroom commitment and the dependent variable is the final test, the statistical results are above. From the obtained statistical data, it can be observed that multiple R is 0.97 which means that the correlation between the independent variable (classroom commitment) and depend variable is relatively close. Also R square is 0.95 which means that 95% of the change of dependent variable is explained by the independent variable.

Table 4. Simple linear regression model for variable X3 (Extra hours)
For the model where extra hours is an independent variable while as a dependent variable is the final test result, multiple R is 0.97 which means that the relationship between the independent variable which is extra hours and depend variable is very close. Also R square is 0.94 which means that 94% of the change of dependent variable is explained by the independent variable that are extra hours and only 0.06 is explained by variables which are not included in the model.

Table 5. Simple linear regression model for variable X4 (Pre-grades)

For the model with pre-grades as an independent variable and final test result as a dependent variable, multiple R is 0.98 which means that the relationship between the independent variable which is pre-grades and dependent variable which is final test is relatively very close. Also R square is 0.97 which means that 97% of the change of dependent variable is explained by the independent variable that are pre-grades and only 0.03 is explained by variables which are not included in the model.

Figures of data analyses from SPSS for class X-7 for independent variables X1 (Homework), X2 (Classroom Commitment), X3 (Extra hours), X4 (Preliminary grades) and final test as a depend variables for simple and multiple linear regression are shown in Appendix, and similar results are obtained as in Excel.
3.1 Interpretation of results for class X- 7 and discussion
Meaning of irrational numbers are essential for expanding and reconstructing the concept of number from the rational number system to the real number system (Kidron, I.,2016). Generally the exam contains exercises of irrational numbers. In general, students have problem generally with irrational numbers but also with the part of rationalization, although the maximum effort was made to explain and make it clear. Maybe the expectations of where they might make mistakes, may have been accurate because part of the rationalization only those who were active doing their homework and regularly attending extra hours did not make mistakes. While those who have not been very regular with homework but also have not shown activities during school hours have encountered difficulties in solving the exercises with rationalization and radicals. The part of rationalization and expressing the radical into an exponent has been explained in class earlier so it would not be much of a problem for students but the poor commitment of students has resulted in unsatisfactory results. From the analyses made using the two software applications we see the same result obtained from the tables. Some of the individual parameters are interpreted as follows:
The multiple correlation coefficient (R) represents the level of the relationship between the dependent variable and two or more independent variables. The closer to the unit, the greater the relationship between the dependent variable and the independent ones. In our results we see that $R > 0.96$ in all cases, which means that the correlation between the variable $Y$ and the variables $X_1, X_2, X_3, X_4$ is relatively strong. $R^2$ is a statistical measure which shows the proportion of variance for the dependent variable which is explained by the independent variables in the regression model. The value $R^2 > 0.93$ or 93% of the change in the dependent variable is explained by the variables of homework, classroom commitment, extra hours and preliminary grades. The remaining 7% is explained by variables which are not included in the model by random error. The value $R^2 > 0.93$ means that the found linear regression models are representative. We also see from the tables that the most representative is the multiple linear regression model. The statistical significance of the regression model is determined based on the values of the empirical ratio F, i.e. the corresponding value $p$, where in the case of the multiple model is $1.8 \cdot 10^{-12}$ with the risk level $p <0.01$ we can say that at least one of the regression variables has an impact on the significance of statistics in the last test in mathematics, respectively multiple linear regression is statistically significant. In the following cases, the statistical significance of regression models is of great value. The data obtained from the extraction of regression tables provide information on the statistical significance of the respective regression coefficients. The statistical significance of the regression coefficients was determined based on the T-test i.e., the corresponding values $P$. In our case this value is less than 0.05, so we can conclude that the alternative $H_A$ hypothesis is
accepted which means that all variables independent have an impact on the dependent variable, therefore in conclusion the linear regression models of the four variables have an impact on the statistical significance "in the test in mathematics".

In terms of homework, the students had a total of five homework assignments, class activities were usually group work with students, i.e. they were given a mathematical problem and were asked to work in groups on the given exercises. There was a very good result working in groups with students. The other activity is the use of the program GeoGebra and Mathematica for solving exercises by students then solving exercises on the whiteboard. X classes students in high school find themselves better when working with GeoGebra, compared to lower classes (Mollakuqe V, Rexhepi S, Iseni E.,2019). All these are part of the activation of students during the lesson in mathematics. The test has been similar to homework normally done with some added requirements that distinguishes the test at least slightly from homework and classroom activities. Tests are usually based on Bloom’s taxonomy. So first is recognition, understanding, implementation, analysis, evaluation and creative work. Below are some exercises solved by the students, while some of the tests are presented in the appendix. The solved exercises are from those students who have worked very well in the test, are from those who have worked moderately in the test and from those who have worked poorly.

**Exercise: Find** $Re(z)$ **and** $Im(z)$ **if the following equalities are given:**

**Solution:**

Figure 1. This exercise is solved by a student who has worked hard in the test
a) 
\[
\text{Re}(z) = (\sqrt{2} + \sqrt{x^3}i) = (\sqrt{2} + x^2i) = \sqrt{2}
\]
\[
\text{Im}(z) = (\sqrt{2} + x^2i) = x^2
\]

b) 
\[
\left(\frac{2}{4+3i}\right) = \left(\frac{2}{4+3i} \cdot \frac{4-3i}{4-3i}\right) = \left(\frac{8-6i}{16-12i+12i-9i^2}\right) = \left(\frac{8-6i}{16+9}\right) = \left(\frac{8-6i}{25}\right) \Rightarrow \text{Re}(z) = \frac{8}{25}, \text{Im}(z) = -\frac{6}{25}
\]

The exercise taken in this test is a homework done both in class and in homework. So it is a frequent exercise and is considered an easy exercise that enters the part of understanding complex numbers. This student has done all the homework and has been active in almost all classes, as well as in extra lessons.

Exercise: Rationalize the following fraction.

Solution:

Figure 2. This exercise was done by a student who worked on average in the test.
\[
\frac{\sqrt{4}}{\sqrt{3} + \sqrt{2}} = \frac{4}{\sqrt{3} \sqrt{2}} \cdot \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{12} - \sqrt{8}}{\sqrt{9} - \sqrt{6} + \sqrt{6} - \sqrt{4}} = \frac{\sqrt{12} + \sqrt{8}}{3 - 2} = \sqrt{12} - \sqrt{8}
\]

This test was performed by a student who worked on average in the test. Also his activity in the classroom has been unsatisfactory and also has not completed all homework. The exercise is done in extra hours in math and in the class so it is similar there may be a change in number or root but it is almost the same, but the student was not present in all extra hours so the exercise is not realized correctly but with some mistakes as seen to have been improved during the test control. In the class and extra lessons it was mentioned to distinguish rationalization, when the fraction has monomial denominator, the numerator and denominator of the fraction is multiplied by same denominator and in binomial denominator, the numerator and denominator of the fraction is multiplied by the conjugate of the denominator in order to use the famous formula \((a-b)(a+b)=a^2-b^2\). These kind of exercises usually requires more to practice and solve homework, that some of students have not done.

**Exercise:**

a) **Turn the radical into exponential.**

b) **Applying the properties of exponentials calculate the value of the following expression.**

**Solution:**

![Figure 3. Exercise done by the student who did poorly in the test.](image-url)

a) \(\sqrt[4]{x^7} = x^{\frac{7}{4}}\)

b) \(2^0 + 3^2 - (-2)^2 = 1 + 9 - 4 = 6\)
This exercise is solved by the student who is not active in the class, did not do all the homework on time, and also in the extra hours did not always participate and therefore in the test showed unsatisfactory success. This exercise is one of the simple exercises which is done by the student because in fact it is an exercise which is part of the recognition because it is foreseen that this part of empowerment and rooting has passed in the IX grade, so they already had known this part, therefore also does not have many points in the test. The exercise is correctly solved by the student but they are not enough to get a good grade classified in students who have worked poorly. this exercise is done in class in homework in activities organized during the lesson and also in extra hours.

As for the test questions, for example, the first question is very simple and is an exercise worked in the classroom, maybe it can just be different numbers. The second exercise is from the given homework. While the last exercises are usually from those similar exercises that have been worked on in the classroom but with different numbers and a little more difficult and are required to be solved by the student.

4. **Data analyzed with Excel from X - 3**

The equation of the multiple linear regression model is:

\[ Y = 21.72 + 1.83X_1 + 7.2X_2 + 9.47X_3 - 4.31X_4 \]

Regression statistics for the multiple regression model are presented in (Table 6).

The regression equation of the simple linear model for the variable \( X_1 \) (homework) is:

\[ Y = 17.29 + 14.59X_1 \]

The regression statistics of the simple linear model for the variable \( X_1 \) is given in (Table 7).

The regression equation of the simple linear model for the variable \( X_2 \) (classroom commitment) is:

\[ Y = 19.57 + 14.30X_2 \]

The regression statistics of the simple linear model for the variable \( X_2 \) is given in (Table 8).
The regression equation of the simple linear model for the variable $X_3$ (extra hours) is:

$$Y = 17.35 + 14.92X_3$$

The regression statistics of the simple linear model for the variable $X_3$ is given in (Table 9).

The regression equation of the simple linear model for the variable $X_4$ (pre-grades) is:

$$Y = 4.57 + 16.47X_4$$

The regression statistics of the simple linear model for the variable $X_4$ is given in (Table 10).

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Table 6. Statistical regression for the multiple regression model.

Here are analyzed the data in Excel for class X-3 where as independent variables are: X1 (homework), X2 (classroom commitment), X3 (extra hours) and X4 (pre-grades) while the dependent variable is the final test. Then from the statistics obtained from the marked data we get that multiple R is 0.98 which means that the correlation between final test and four these
independent variables is relatively strong. Also R Square is 0.97 which means that 97% of the change of depend variable is explained by independent variables of homework, classroom commitment, extra hours and preliminary grades and 0.03 is explained by variables which are included in the model by random error which we explained above.

![Summary Output Table]

Table 7. Statistical regression for the simple regression model for variable X1 (Homework)

Data analyzed in Excel in which the independent variable is homework and dependent variable is the final test the statistical results are provided in Table 7 above. From the obtained statistical data it can be noticed that multiple R is 0.96 which means that the correlation between the independent variable (homework) and depend variable is relatively close. Also R square is 0.92 which means that 92% of the change of dependent variable is explained by the independent variable.
Table 8. Statistical regression for the simple regression model for variable X2 (Classroom commitment).

Here are the data analyzed in Excel which the independent variable is classroom commitment and dependent variable is the final test. From the obtained statistical data we see that multiple R is 0.97 which means that the correlation between the independent variable (classroom commitment) and depend variable which is final test is relatively close. Also R square is 0.94 which means that 94% of the change of dependent variable is explained by the independent variable and only 6% is explained by variables which are not included in the model.
Table 9. Statistical regression for the simple regression model for variable X3 (Extra hours).

Data analyzed in Excel which the independent variable is extra hours and dependent variable is the final test. From the obtained statistical data we see that multiple R is 0.98 which means that the correlation between the independent variable (extra hours) and depend variable which is final test is relatively strong. Also R square is 0.96 which means that 96% of the change of dependent variable is explained by the independent variable and only 4% is explained by variables which are not included in the model.
Table 10. Statistical regression for the simple regression model for variable X4 (Pre-grades).

Here are the data analyzed in Excel which the independent variable is pre-grades and dependent variable is the final test. From the obtained statistical data we see that multiple R is 0.94 which means that the correlation between the independent variable (pre-grades) and dependent variable which is final test is relatively close. Also R square is 0.88 which means that 88% of the change of dependent variable is explained by the independent variable and only 12% is explained by variables which are not included in the model.

Figures of data analyses from SPSS for class X-3 for independent variables X1 (Homework), X2 (Classroom Commitment), X3 (Extra hours), X4 (Preliminary grades) and final test as a dependent variable for simple and multiple linear regression are shown in Appendix, and similar results are obtained as in Excel.

4.1 Interpretation of results for class X-3

The test generally contains tasks of irrational numbers. In general, students have a problem with this part of irrational numbers even though they learned this part of irrational numbers and rationalization in the previous grade. However, watching and observing the students during the class, it was expected that they could make mistakes in the part of rationalization, therefore
students were given enough exercises to practice and almost every student is activated to solve such an exercise, then normally that the expectations have been not to make mistakes. We know that not everything depends on us teachers, our part as teachers is to give students and explain the meaning of the methods of what are irrational numbers, but also a large percentage belongs to students to engage in school and in homes to be more demanding and have the proper preparation when undergoing the final test. From the analyzes made using the two software applications we see the same result obtained from the tables. Some of the individual parameters are interpreted as follows: The multiple correlation coefficient ($R$) represents the level of the relationship between the dependent variable and two or more independent variables. The closer to the unit, the greater the relationship between the dependent variable and the independent ones. In our results we see that $R > 0.94$ in all cases, which means that the correlation between the variable $Y$ and the variables $X_1, X_2, X_3, X_4$ is relatively strong. $R^2$ is a statistical measure which shows the proportion of variance for the dependent variable which is explained by the independent variables in the regression model. The value $R^2 > 0.88$ or 88% of the change in the dependent variable is explained by the variables of homework, classroom commitment, extra hours and preliminary grades. The remaining 12% is explained by variables which are not included in the model by random error. The value $R^2 > 0.88$ means that the found linear regression models are representative. We also see from the tables that the most representative is the multiple linear regression model. The statistical significance of the regression model is determined based on the values of the empirical ratio $F$, i.e. the corresponding value $p$, where in the case of the multiple model is $6.62 \cdot 10^{-10}$ with the risk level $p < 0.01$ we can say that at least one of the regression variables has an impact on the significance of statistics in the last test in mathematics, respectively multiple linear regression is statistically significant. In the following cases, the statistical significance of regression models is of great value. The data obtained from the extraction of regression tables provide information on the statistical significance of the respective regression coefficients. The statistical significance of the regression coefficients was determined based on the $T$-test i.e., the corresponding values $P$. In our case this value is less than 0.05, so it can be concluded that the alternative $H_A$ hypothesis is accepted which means that all variables independent have an impact on the dependent variable, therefore in conclusion the linear regression models of the four variables have an impact on the statistical significance "in the test in mathematics".

5. CONCLUSIONS AND RECOMMENDATIONS

From the above analyzes, our multiple regression model as well as simple linear regression models to predict and analyze the test in the Mathematics subject was adequate and usable. As a result, we saw that the model built above is statistically significant, because based on the analysis we did in
relation to the model we saw that what we wanted to prove was verified with the help of these statistical programs such as Excel and SPSS. So in conclusion this model is of great importance in the mathematics methodology where it shows that independent variables such as: homework, classroom commitment, extra hours and pre-grades have an impact on test results so we can predict from independent variables that what can be expected from students in the test. This built model is not exhaustive except for these independent variables that we mentioned above we can take different examples and analyze them through these statistical programs such as we can look at the methods we use when teaching in the classroom, compare which of them has the greatest impact on school learning and recommend these methods since these methods achieved to increase the success of the students and improve the quality of education in mathematics. The analyzes which have been done on the components mentioned in the paper (homework, activity, etc.) have enabled these components to be presented during the lesson and this has given a positive result and impact to students and this method has reflected in them in grading in the second semester.

We have also reached to a conclusion that critical thinking should be applied to students. All students have the ability to grow and develop critical thinking skills when learning math. For example, during this research in one of the planned hours were complex numbers. As usual at the beginning of the class has been used the brainstorming method. Students were asked the question: What do they know about complex numbers? Normally, we have received various answers that are not necessarily correct but that have given their critical opinion to the question. Critical thinking and reasoning allow students to think about how they utilize their discipline of mathematical skills. Students can develop this ability when confronted with mathematical problems, reaching possible solutions to those mathematical problems, and evaluating and justifying the results obtained, thus allowing students to become confident critical thinkers. Students have had different independent critical thinking about homework, activity, extra hours, etc. In general, there were positive thoughts, the reasons were that the students were constantly in the course of lessons, their activities in the classroom were evaluated, which means that the student was more active because it influenced them to get better grades. Some students were honest and admitted that the poor result in the test was as a result of not doing homework, absence in extra-hour lessons or their lack of interest during the activity in the classroom. While in our opinion the importance of these activities mentioned in the paper is extremely high, which was supported and by statistical methods. For each student we have to keep records every time because the assessment in the end will be more accurate and fairer and the possibility of error will be smaller.
REFERENCES


APPENDIX

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SPSS data analysis for class X - 7

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a. Predictors: (Constant), X1

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b. Predictors: (Constant), X1

c. Predictors: (Constant), X2

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b. Predictors: (Constant), X2

c. Predictors: (Constant), X2

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Table 11. Simple linear regression model for variable $X_1$, for class $X - 7$

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**ANOVA**

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a. Dependent Variable: $Y$

b. Predictors: (Constant), $X_3$

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Table 12. Simple linear regression model for variable $X_2$, for class $X-7$

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a. Predictors: (Constant), $X_4$

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Table 13. Simple linear regression model for variable $X_3$ for class $X-7$,

Table 14. Simple linear regression model for $X_4$ for class $X-7$,

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http://www.hostos.cuny.edu/mtrj/
## Model Summary

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a. Predictors: (Constant), X4, X3, X2, X1

## ANOVA<sup>a</sup>

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a. Dependent Variable: Y
b. Predictors: (Constant), X4, X3, X2, X1

## Coefficients<sup>a</sup>

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a. Dependent Variable: Y

Table 15. Multiple linear regression model for class X - 7
Data analysis with SPSS for class X-3

Table 16. Simple linear regression model variable X1, for class X - 3.

Table 17. Simple linear regression model for variable X2, for class X - 3.
Table 18. Simple linear regression model for variable X3, for class X - 3

Table 19. Simple linear regression model for variable X4, for class X - 3
Table 20. Multiple linear regression model for class X – 3

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<th>R Square</th>
<th>Adjusted R Square</th>
<th>Std. Error of the Estimate</th>
</tr>
</thead>
<tbody>
<tr>
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<td>.972</td>
<td>.963</td>
<td>5.04036</td>
</tr>
</tbody>
</table>

\(^a\) Predictors: (Constant), X4, X1, X3, X2

**ANOVA\(^a\)**

<table>
<thead>
<tr>
<th>Model</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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<td>4</td>
<td>2814.558</td>
<td>110.787</td>
<td>&lt;.001(^b)</td>
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<tr>
<td></td>
<td>Residual</td>
<td>13</td>
<td>25.405</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>17</td>
<td>11588.500</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) Dependent Variable: Y

\(^b\) Predictors: (Constant), X4, X1, X3, X2

**Coefficients\(^a\)**

<table>
<thead>
<tr>
<th>Model</th>
<th>Unstandardized Coefficients</th>
<th>Standardized Coefficients</th>
<th>t</th>
<th>Sig.</th>
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</thead>
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<td>B</td>
<td>Std. Error</td>
<td>Beta</td>
<td>t</td>
</tr>
<tr>
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<td>(Constant)</td>
<td>21.722</td>
<td>3.969</td>
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<tr>
<td></td>
<td>X1</td>
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<tr>
<td></td>
<td>X2</td>
<td>7.202</td>
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</tr>
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<td></td>
<td>X3</td>
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<td></td>
<td>X4</td>
<td>-4.312</td>
<td>3.554</td>
<td>-.246</td>
</tr>
</tbody>
</table>

\(^a\) Dependent Variable: Y
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http://www.hostos.cuny.edu/mtrj/
Some test results from the students
Mathematical competence in preschool students and its relationship with intelligence, age and cognitive functions of attention, information processing speed and reaction inhibition

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Abstract: The purpose of this study is to examine the relationship between flow intelligence (Gf), age and cognitive abilities of processing speed (Comp) of attention (Incomp) and reaction inhibition (Flef) in relation to the level of Mathematical proficiency (MP) of preschool students. Sixty-four kindergarten students participated in the research. Based on the results, it was shown that mathematical competence (MP) shows a strong positive correlation with the variables "age" (age) "flow intelligence" (Gf). These variables were found to be able to predict the level of mathematical competence. The variables "information processing speed" (Comp), "attention" (Incomp) and "reaction inhibition" (Flef) do not seem to be significantly related nor can they predict the level of mathematical competence. These findings can be used both by teachers (e.g. implementation of early intervention programs) and by those responsible for planning and formulating educational policy.

Keywords: Mathematical competence, flowing intelligence, age, attention, processing speed, reaction inhibition.

INTRODUCTION

There are two approaches to mathematical performance. One approach considers that number recognition and estimation, comparison skills, comprehension, and measurement ability are the determinants of predictive mathematical performance (Geary, Hamson & Hoard 2000; Toll & Van Luit, 2013). These skills represent knowledge and experience gained through contact with the
environment (social, cultural, educational) and constitute what is called crystallized intelligence (Gc) (Horn & Cattell, 1967) and require complex cognitive functions.

The second approach estimates that a number of cognitive functions, such as working memory, processing speed, attention, response inhibition (Fias, Menon & Szucs, 2013; Namkung & Fuchs, 2016) have shown a direct relationship to the development of mathematical skills in a wide range of ages (Bull, Espy & Weibe, 2008). However, mathematical performance and its relationship to skills and cognitive functions have not been substantially examined in many critical early areas of mathematics, while most studies usually focus on only one mathematical area.

**LITERACY REVIEW**

Mathematical performance research provides important information for understanding mathematical difficulties and the factors that contribute to it. Intelligence is an important factor and is associated with a number of cognitive skills, such as processing speed (Fry & Hale, 2000). Some researchers estimate that the effect of processing speed on intelligence is indirect and not direct (Fry & Hale, 1996) while others consider that it affects the performance of intelligence tests directly and indirectly (Kail, 2000). Some researchers (Demetriou et al., 2013) estimate that the power of the relationship between processing speed and intelligence varies with age. Others (Fry & Hale, 2000) argue that there is no systematic change with age in the size of the correlation between processing speed and flowing intelligence.

An attempt has been made to determine the relationship between intelligence and attention. Heitz, Unsworth and Engle (2005) point out that attention is one of the determinants of flowing intelligence. Research results support the correlation between attention and intelligence (Burns, Nettelbeck & McPherson, 2009). But there is very little evidence of this relationship in the child population. Völke & Roebers (2016) report that while working memory and flowing intelligence are significantly related, continuous attention is not directly related to flowing intelligence and working memory in childhood.

Inhibition of reaction as a factor is of great interest because it relates to academic performance (Gottfredson, 1997a, b, 1998). An important issue is the development of inhibition mechanisms and their relationship to the development of intelligence. Studies of school-age children with
attention deficit hyperactivity disorder (ADHD) - who have an inability to react due to the disorder - and of normal developmental students have not provided sufficient data linking intelligence to reaction inhibition (Bitsakou et al., 2008; Oosterlaan & Sergeant, 1996; Rubia et al., 1998). But other researchers believe that the two abilities are closely related (Dempster, 1991). In general, there is a correlation between reaction inhibition and flow intelligence but this correlation seems to be small (Michel & Anderson, 2009). Other results show that there is no correlation (Arán-Filippetti, Krumm & Raimondi, 2010).

Research data show that the relationships between different variables are many and also quite complex. For example, processing speed is an important factor associated with academic achievement in childhood and appears to be strongly correlated with attention function (Colom et al, 2008). However, measuring response time to calculate processing speed used in attention studies (de Kieviet et al., 2012) may not be a reliable means of evaluation. Special attention should be taken not to confuse slow processing speed with the Slow Cognitive Tempo (SCT) phenomenon which is a complex of symptoms involving inconsistent alertness and orientation and is characterized by slowness, drowsiness and daydreaming (McBurnett, Pfiffner & Frick, 2001).

Attention and response inhibition are very basic processes, and are related to cognitive processing and are often taken and classified as almost identical (Barkley, 1997). Some researchers attempts have been made to study the relationship between these two cognitive processes, but there are many things to be understood yet. The two functions appear to have the same functional and structural aspects and similar evaluation tests. The similarity of these two functions is so close that researchers often assume that they are two sides of the same coin (Mostofsky & Simmonds, 2008).

Regarding mathematical performance, studies have investigated its correlation with basic processes and factors. Thus, the relationship between mathematical performance, age, intelligence, processing speed, attention, and response inhibition was tested. The age factor and its relationship with academic performance have occupied both researchers and those responsible for designing and formulating educational policy. For example in a kindergarten class some children may be five years old (60 months) while their classmates in the same class may be much older (70 months). Potential differences in performance may be due to the age difference. Studies (Yesil, Dagli & Jones, 2012; Crawford, Dearden & Greaves, 2014) have shown that older children showed better math skills than their younger classmates.
Intelligence is one of the most studied predictors of mathematical performance. The data show that individual differences in mathematical performance are related to intelligence. Characteristically, Cattell (1987) states: “This year's level of competence is a function of the level of current intelligence and last year's interest in schoolwork.” The strength of the relationship between intelligence and mathematical performance varies depending on the class, the application of innovations, and the degree of difficulty and complexity of the field of mathematics (Fuchs et al., 2010).

Processing speed is a central mental ability that pushes changes in knowledge to a higher level. High processing speed seems to be associated with higher academic performance. Findings show that children with poorer mathematical performance are slower at processing information (Bull & Johnston, 1997; Geary et al., 2012). Processing speed seems to be closely related to other cognitive functions. Thus it is extremely difficult to answer questions such as whether individual differences in working memory are due to fundamental differences in the speed of cognitive processing and decision making or whether attention accelerates information processing.

Attention is a basic, though less complex than others, cognitive ability, the role of which in relation to the school context is examined. Research on the predictive validity of attention to academic performance provides mixed results. Colom, Escorial, Shih, and Privado (2007) reported only very low and insignificant correlations ($r < 0.20$) between attention and school performance. But another study (Luo, Thompson & Detterman, 2006) showed that basic cognitive processes such as attention are good predictors of school performance. However, the level of attention is probably affected by other cognitive skills, such as executive functions, while the assessment of attention is difficult. The results of intervention training programs vary depending on the type of population being treated and the age. It seems that such programs are more effective in older children with attention deficits and in younger children (Peng & Miller, 2016).

In recent years there has been an increase in studies that have investigated the role of response inhibition in mathematical performance. The majority of studies investigate the degree of correlation between these two factors. For example, performance in response inhibition tests is related to performance, both in informal assessment procedures (Visu-Petra et al., 2011) and in standardized mathematical tests (St Clair-Thompson & Gathercole 2006). However, other research data (Waber et al., 2006) found weak relationships between response inhibition and mathematical performance and there are studies that could not find evidence for the relationship between
response inhibition skills and mathematics. Monette et al. (2011) found that response inhibition predicted future performance in reading but not in mathematics.

PURPOSE OF THE STUDY

The purpose of this study is to examine the relationship between mathematical competence with mental ability, age, attention, processing speed and response inhibition. Specifically, what are the relationships between attention, information processing speed, response inhibition, age and mental abilities and what is their relationship with mathematical competence?

Based on the above theoretical and research data, we were led to formulate the research questions:

- Are the variables, mathematical competence, flowing intelligence, age, attention, processing speed, response inhibition related to each other?
- The variables, flowing intelligence, age, attention, processing speed, response inhibition are related to the individual scales of the Utrecht Mathematical Adequacy Criterion and to what extent in the age group of kindergarten students?
- Can cognitive skills, age and flowing intelligence be predictors of mathematical performance and at what level?

METHODOLOGY

Participants

The study involved 64 kindergarten students (36 boys and 28 girls) with an average age of 5 years and 7 months (SD = 5.60 months, min = 56 months, max = 76 months). All students completed the test activities.

Procedure

All participants were assessed by researchers at their school individually during school hours at the end of the school year. The consent of their parents was sought for the whole procedure. The assessment was done individually by the research team in a quiet room of the school at the end of
the school year, in the months of May and June. The students who participated in this study were rewarded for their participation and therefore committed to perform the activities correctly.

**Measurements**

The tools used to measure the cognitive abilities and flowing intelligence (Gf) of the sample students are described below. The children completed their assessment in two phases. In the first phase, flowing intelligence (Gf) and cognitive skills of attention, processing speed and response inhibition were assessed. In the second phase, the degree of mathematical competence was assessed. Children were allowed short breaks during the assessment to ensure optimal assessment conditions.

**Flowing intelligence**

Flowing intelligence (Gf) has been found to be highly related to general intelligence and has a prominent place in studies on academic performance. Flowing intelligence (Gf) was assessed using Raven Colored Progressive Matrices (CPM), which requires analytical reasoning for abstract audiovisual material. This test is known to be one of the most important tests for assessing flowing intelligence. Its instructions are simple and its implementation takes little time. Students were asked to choose the correct one for each item on the answer sheet. Silence and individual work were necessary. The maximum score was 36.

**Mathematical competence**

Early mathematical competence refers to the overall level of knowledge and skills required to effectively introduce a preschool and early school child to formal school mathematics (Van de Rijt, Van Luit & Pennings, 1994). The Utrecht Early Mathematical Competence Criterion was chosen for the assessment of early mathematical competence. This is a single test consisting of eight scales of tests - questions. The scale of the criterion covers both Piaget's skills (comparison, classification, matching and serialization) and counting skills. This criterion is used to assess the level of development of early arithmetic ability. The eight sections of the criterion are: Comparison, classification, matching, serialization, use of numbering words, structured counting, effective counting, general knowledge of numbers.

**Attention, processing speed, response inhibition.**
The cognitive functions of attention, processing speed and response inhibition were measured by the Eriksen Flanker Task test (Eriksen & Eriksen, 1974). Compatible reaction times in Stroop, Simon and Eriksen's works are used to measure information processing speed, while incompatible reaction times are used to measure processing control. In the above tests the reaction times to the compatible condition are shorter than the reaction times to the incompatible condition. The difference in performance between Compatible and incompatible conditions is called the “Flanker effect” and is considered to measure the inhibition of the reaction. In other words, it measures the ability to inhibit irrelevant competing responses to a non-verbal stimulus (Eriksen & Eriksen, 1974).

RESULTS

Descriptive statistics

The averages, standard deviations, performance range, lowest and highest performance per assessment category are presented in the table below (Table 1). Based on the data of the statistical table, it is observed that the average mathematical performance (MP) of the sample was 29.71 (Std. D = 7.3), the average age was 66.68 months, the performance in the measuring tool of flowing intelligence (Gf) was 19.21 (Std. D = 4.65). The average performance in measuring the processing speed (Comp) was 1209.3 (Std. D = 174.54), the average performance in attention control (Incomp) was 1220.28 (Std. D = 188.35) and the average performance in the response inhibition test (Flef) was 12.59 (Std. D = 137.74).

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation (Std.D)</th>
<th>Sample (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MP</td>
<td>29.7188</td>
<td>7.39040</td>
<td>64</td>
</tr>
<tr>
<td>age</td>
<td>66.6875</td>
<td>5.61425</td>
<td>64</td>
</tr>
<tr>
<td>Gf</td>
<td>19.2188</td>
<td>4.65123</td>
<td>64</td>
</tr>
<tr>
<td>Comp</td>
<td>1209.3125</td>
<td>174.54584</td>
<td>64</td>
</tr>
<tr>
<td>Incomp</td>
<td>1220.2813</td>
<td>188.35297</td>
<td>64</td>
</tr>
<tr>
<td>Flef</td>
<td>12.5938</td>
<td>137.74761</td>
<td>64</td>
</tr>
</tbody>
</table>
MP = Maths performance, Age, Gf = flowing intelligence, Comp = processing speed, Incomp = attention control, Flef = response inhibition

Table 1: Descriptive Statistics

The Pearson correlation coefficient was calculated to examine the correlations between the areas of assessment (Figure 1).

![Correlation Diagram]

MP = Maths performance, Age, Gf = flowing intelligence, Comp = processing speed, Incomp = attention control, Flef = response inhibition

Figure 1: Correlations between variables

The above figure (figure 1) also shows the correlation of variables with mathematical competence (MP). The results show that mathematical competence (MP) has a strong positive correlation with age (r = .522) and flowing intelligence (Gf) (r = .601), while the correlation with processing speed (Comp) (r = .075), attention (Incomp) (r = .146) and reaction inhibition (Flef) (r = .111) is weak.

However, in order to be able to answer the second research question concerning the correlation of the individual scales of the Utrecht mathematical criterion (comparison - Com, classification Ci, matching - mat, serialization - seq, use of numbering words - Numw, structured counting - Str ,
effective counting - Efco, general knowledge of numbers - Knon) are related and to what extent with variables, flow intelligence (Gf), age (age), attention (Incomp), processing speed (Comp), reaction inhibition (Flef). The results are presented in Table 2.

<table>
<thead>
<tr>
<th></th>
<th>Age</th>
<th>Gf</th>
<th>Comp</th>
<th>Incomp</th>
<th>Flef</th>
</tr>
</thead>
<tbody>
<tr>
<td>Com</td>
<td>-.154</td>
<td>.096</td>
<td>-.136</td>
<td>-.181</td>
<td>-.072</td>
</tr>
<tr>
<td>Ci</td>
<td>.251</td>
<td>.460</td>
<td>.144</td>
<td>.369</td>
<td>.330</td>
</tr>
<tr>
<td>Mat</td>
<td>.474</td>
<td>.513</td>
<td>-.083</td>
<td>-.010</td>
<td>.093</td>
</tr>
<tr>
<td>seq</td>
<td>.306</td>
<td>.313</td>
<td>.055</td>
<td>.024</td>
<td>-.028</td>
</tr>
<tr>
<td>Numw</td>
<td>.350</td>
<td>.578</td>
<td>.065</td>
<td>.187</td>
<td>.170</td>
</tr>
<tr>
<td>Strc</td>
<td>.534</td>
<td>.585</td>
<td>.179</td>
<td>.108</td>
<td>-.067</td>
</tr>
<tr>
<td>Efco</td>
<td>.482</td>
<td>.385</td>
<td>.086</td>
<td>.216</td>
<td>.195</td>
</tr>
<tr>
<td>Knon</td>
<td>.456</td>
<td>.506</td>
<td>.002</td>
<td>.018</td>
<td>.025</td>
</tr>
</tbody>
</table>

Com= comparison, Ci= classification, Mat= matching, seq= Sequencing, Numw= use of number words, Strc= structured count, Efco= effective count, Knon = General knowledge of numbers, MP= mathematical competence, age= age, Gf= flowing intelligence, Comp= processing speed, Incomp= attention, Flef= response inhibition

Table 2: Correlations of the variables

Based on these results (Table 2) it seems that all subscales (except the comparison subscale - Com) of the mathematical competence test are moderately to strongly related with the variables age (Gf) and flowing intelligence (Gf) while they are very weakly related or not at all with the variables attention (Incomp), processing speed (Comp) and response inhibition (Flef). An exception is the classification - (Ci) which shows that it is the only one of the subscales of the mathematical competence criterion that shows a moderate correlation with the variables processing speed - (Comp) (r = .369) and response inhibition - (Flef) (r = .330).

In order to answer the third research question, that is which of the variables we examine can contribute to the level of mathematical competence, a regression analysis was performed. It was
chosen to enter all the variables at the same time to determine which of the defined variables contribute to the mathematical competence and how much variation they explain. During the adjustment test (table 3 – model summary) it seems that the model explains 47.6% of the total variability (R Square = .476) while the Adjusted R Square index is .431. The comparison of the two indicators (R Square and Adjusted R Square) shows that the findings can be generalized to the population.

<table>
<thead>
<tr>
<th>Model</th>
<th>R</th>
<th>R Square</th>
<th>Adjusted R Square</th>
<th>Std. Error of the Estimate</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>.690a</td>
<td>.476</td>
<td>.431</td>
<td>5.57448</td>
</tr>
</tbody>
</table>

a. Predictors: (Constant), Flef, Gf, Comp, Age, Incomp

Table 3: Model Summary

Regarding the significance of the model (Table 4) we observe that the F-test shows a significance of sig <0.000 so the model is very important for explaining variability and contributes significantly to the prediction of mathematical competence.

<table>
<thead>
<tr>
<th>Model</th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>1638,599</td>
<td>5</td>
<td>327,720</td>
<td>10,546</td>
<td>.000b</td>
</tr>
<tr>
<td>Residual</td>
<td>1802,339</td>
<td>58</td>
<td>31,075</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>3440,938</td>
<td>63</td>
<td>61,075</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Dependent Variable: MP
b. Predictors: (Constant), Flef, Gf, Comp, Age, Incomp

Table 4: Significance of the model (ANOVA)

The results on the degree of contribution of the variables to mathematical competence are presented in table 5. Regarding the parameters of the model as shown (table 5) the coefficients of the variables are positive while only the variable attention (Incomp) has a negative value - .161. The Unstandardized Coefficient is .370 for the variable "age", and .792 for the "flowing intelligence" (Gf ). Therefore for each increase of these variables by one unit the effect on the
increase of mathematical competence is respectively, 370 from the contribution of the increase of age (Sig. = .025) and .792 from the contribution of the flow of intelligence, (Sig. = .025).

<table>
<thead>
<tr>
<th>Model</th>
<th>Unstandardized Coefficients</th>
<th>Standardized Coefficients</th>
<th>t</th>
<th>Sig.</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Beta</td>
<td></td>
<td></td>
<td>Zer-order</td>
</tr>
<tr>
<td>(Constant)</td>
<td>-16.159</td>
<td>11.847</td>
<td>-1.364</td>
<td>.178</td>
<td></td>
</tr>
<tr>
<td>age</td>
<td>.370</td>
<td>.161</td>
<td>.281</td>
<td>2.306</td>
<td>.025</td>
</tr>
<tr>
<td>Gf</td>
<td>.792</td>
<td>.187</td>
<td>.498</td>
<td>4.236</td>
<td>.000</td>
</tr>
<tr>
<td>Comp</td>
<td>.166</td>
<td>.105</td>
<td>3.919</td>
<td>1.586</td>
<td>.118</td>
</tr>
<tr>
<td>Incomp</td>
<td>-.161</td>
<td>.106</td>
<td>-4.109</td>
<td>-1.523</td>
<td>.133</td>
</tr>
<tr>
<td>Flef</td>
<td>.162</td>
<td>.106</td>
<td>3.014</td>
<td>1.525</td>
<td>.133</td>
</tr>
</tbody>
</table>

a. Dependent Variable: MP

Table 5: Coefficients

**DISCUSSION**

The aim of this study was to examine the variables related to the level of mathematical competence of students at the end of kindergarten, before attending primary school. The possibility of predicting the variables "flowing intelligence" (Gf), "age", "attention" (Incomp), "information processing speed" (Comp), and "reaction inhibition" (Flef) for mathematical competence (MP) was also examined. In addition to the correlation of the level of mathematical competence (according to the mathematical competence criterion of Utrecht) there was a detailed correlation of the individual scales of the criterion of mathematical competence of Utrecht with the factors "flowing intelligence" (Gf), "age", and the cognitive functions "attention" (Incomp), "Information Processing Speed" (Comp) and "Reaction Inhibition" (Flef). This was done to investigate the possible correlations between these variables and the individual scales of the Utrecht mathematical competence criterion. In other words, a more detailed investigation was carried out in order to show correlations that may not be seen from the overall degree (score) of performance of the Utrecht mathematical competence criterion. For example, the degree of final performance in the
mathematical competence criterion, which is the result of overall performance on all eight scales, may not be related to the variable "attention". However, we cannot rule out the possibility that some of the eight scales of the mathematical competence criterion, which constitute the overall degree of performance of the criterion, are related to the variable "attention".

Based on the results (Figure 1) the first question of the research is answered (if and to what extent the variables are related to each other). We observe that the variable mathematical competence (MP) is significantly related to the variables "flowing intelligence" (Gf) \((r = .601)\) and "age" \((r = .522)\). These results are in line with research findings that show a strong correlation between mathematical performance (MP) and "flowing intelligence" (Gf) (McGrew & Wendling, 2010). The results of the present study also confirm the results of those studies that show a strong correlation between mathematical performance and age. It has been observed that older children score higher in mathematics (Crawford, Dearden, & Greaves, 2014; Yesil Dagli & Jones, 2012). The "age" factor is positively correlated to a moderate degree \((r = .480)\) with the "flowing intelligence". The research findings are in line with the results of research showing that flowing intelligence increases into early adulthood (Ackerman, 1996; McArdle et al., 2000).

Variable "processing speed" (Comp) is significantly related to variable “attention” (Incomp) \((r = .712)\). Research findings show that information processing speed and attention are strongly correlated. In a research study (com et al, 2008) it was shown that the two cognitive functions are positively related to each other to a large extent. In fact, “processing speed” was found to explain a large percentage of attention span (McAuley & White, 2011). Also, processing speed tests are able to predict the level of attention, because processing speed requires activation and attention resources (Diamond, 2002).

Variable “reaction inhibition” (Flef) is positively related to “attention” (Incomp) \((r = .471)\). The results of the present study are confirmed by the results of research that show that the two functions are so closely related that researchers often consider that they are essentially two sides of the same coin (Mostofsky & Simmonds, 2008).

Particularly important is the fact that in the present study not only the skills and cognitive functions in relation to mathematical competence were examined but also their relationship with the individual scales of the Utrecht mathematical competency criterion was examined to identify any correlations (second research question).
We do not know of any other research that has examined a correlation of this kind in this age group. The results (Table 2) showed that all sub-scales of Utrecht's mathematical proficiency testing tool (except the comparison - Com) are moderately to strongly related to the variables “age” and “flowing intelligence” (Gf) and very weakly or not at all with the variables "attention" (Incomp), "processing speed" (Comp) and "reaction inhibition" (Flef). Of the eight scales of the mathematical competence criterion, the "classification" scale (Ci) seems to be the only one that shows a moderate correlation with the variables "processing speed" (Comp) (r = .369) and "reaction inhibition" (Flef) (r = .330). The results show that the individual scales of mathematical competence have to a large extent the same tendency of correlation as the overall degree (score) of performance of the Utrecht mathematical competency criterion with the variables “intelligence”, “age”, “processing speed”, “attention”, “reaction inhibition”.

Predicting future performance when a child attends kindergarten is very important. Based on research data, mathematical competence in kindergarten can predict the level of mathematical performance while attending primary school (Jordan, Kaplan, Ramineni & Locuniak, 2009). For this reason, it is extremely useful to know the factors that contribute and can predict the level of mathematical competence (third research question). Based on the results of this research, "flowing intelligence" (Gf) and "age" (age) are strongly related to the level of “mathematical competence” (MP) and can predict it. The results of the present study are in line with the findings of other research (Ferrer & McArdle, 2004; Manginas, Nikolantonakis & Papageorgiou, 2017) which show that some cognitive abilities such as working memory (wm) and flow intelligence (Gf) in addition to their correlation with mathematical performance can also contribute to being the main predictors of academic and mathematical performance. Some findings even show that flow intelligence is a very powerful predictor of mathematical performance and even stronger than that of age (Green, Bunge, Chiongbian, Barrow & Ferrer, 2017). It seems that the basic reasoning skills, which are characteristic of flow intelligence, can predict the acquisition of mathematical skills.

In the present study, the relationship between “age” and “mathematical competence” (older children performed better than younger ones), shows that “age” is another important predictor of “mathematical competence” (MP), a finding that is confirmed by previous studies (Jordan et al., 2006). The better performance and faster development (between first and third grade of elementary school) of older children's math ability can be predicted by the difference in math ability in kindergarten. In fact, the differences in cognitive development between younger and older children...
seem to be there before they start school education (Musch & Grondin, 2001). This is likely because older children (a) are simply older and on average more mature (Bedard & Dhuey, 2008; Stipek, 2002), and (b) have received more experience and support (Gold et al., 2012). Therefore, younger children in kindergarten may have a lower level of maturity in a number of factors and cognitive abilities that in turn affect mathematical performance. After entering the formal education system, these differences still exist or even increase, because the curricula in each class are aimed at high-achieving students and do not favor younger students (Elder & Lubotsky, 2009). However, the attitude of teachers towards students is also different, as teachers have more expectations from the older students in the class, they motivate the older students more, which has a positive learning effect on them, showing a steeper learning curve in them (Stipek, 2002).

The findings of the present study demonstrate the importance of mathematical competence in kindergarten in determining the developmental course of students in primary school mathematics. A seemingly small problem in preschool can cause big problems as the child grows (Karmiloff-Smith, 1998). If children finish kindergarten with a low level of certain abilities and skills, they will go to the next level of education from a disadvantaged position and may never reach the level of children who start primary school with a good level of skills and abilities. Therefore, defining the impact and role of skills and cognitive functions can help in the design and implementation of successful early educational intervention programs that will aim to improve specific skills and functions that positively affect mathematical performance. In this context, the results of the present research can be very useful, especially if we take into account the fact that cognitive skills activities such as reasoning and mental work activities are predictors of academic performance (Deary, Strand, Smith, & Fernandes, 2007; Detterman, 2014a,b; Rindermann & Neubauer, 2004; Schmidt, 2017). It is estimated that the skills practiced through cognitive training programs affect a wide range of areas (Taatgen, 2016). This is especially important for pre-school education because factors such as flowing intelligence and cognitive functions are quite flexible at an early age and can be improved through specialized intervention programs. Flowing intelligence for example has been shown to improve through appropriate educational activities, such as through non-verbal reasoning games (Au et al., 2015). Based on the results of the present research, the factor "flowing intelligence" is particularly decisive for the level of mathematical competence. Therefore, preschool teachers need to pay special attention to this factor and plan and implement activities and programs that will enhance basic reasoning skills. This will have a positive impact on the level of operation of the flowing intelligence. For example, the inclusion of young students in the
kindergarten program and the involvement of board games have a positive effect on the level of abstract and comparative ability that are key elements of mental function. Starting at this age with the "classic" bricks and children's puzzles, the child can develop basic reasoning skills. Games of this type reinforce logic, enhance correlation, enhance the identification of essential elements for problem-solving, and have a positive effect on the level of mental ability. Also, engaging in appropriate digital games has been found to have a positive effect on the level of mental function (Manginas & Nikolantonakis, 2017, 2018). Therefore digital games could be included in the program of activities of the kindergarten students. So in addition to pleasure, enjoyment and satisfaction can be a means of improving the level of mental function. But also engaging with music can have a positive effect on the level of mental function. Especially when the young child learns a musical instrument, his mental skills are strengthened. Research data show a strong correlation between musical audiation and mathematical performance (Manginas, Nikolantonakis, Gounaropoulou, 2018). Mathematics is an activity of organizing and solving problems. However, learning music requires (like mathematics), the organization of information, the creation of structures, and the solution of problems (Pogonowski, 1987). Learning music pushes the student to try to discover different patterns and create structures (Gopnik et al., 2004). But also learning foreign languages seems to be able to contribute positively to the level of mental function (Pimsleur, 1968). Therefore, targeted actions, such as those mentioned, contribute to the increase of the level of mental function. According to the findings of the present study, this will have a positive impact on mathematical performance.

Comparing the relative effects of age differences between students provides important information to teachers. Different levels of children's skills are related to age, which is a factor that is often cited by kindergarten teachers as an obstacle to the implementation of effective teaching (Rimm-Kaufman et al., 2000; Vecchiotti, 2001). The age of the child entering school is also a factor in creating beliefs in teachers about the performance of students (young age is to blame for poor performance of the student) which can have consequences for the child's experiences at school. But parents also systematically identify age as one of the most important dimensions of their view of school readiness (Brent et al., 1996; West et al., 1993). The view of some (Uphoff & Gilmore, 1986) is correct that older children will be more prepared than younger children, so that the increasingly demanding kindergarten curriculum can be fully utilized. But there is also the view that the duration of study in an educational environment is more important than biological maturation and children should be given the opportunity to benefit from school (Stipek, 2002). In
addition, school attendance also functions as a compensation mechanism, which is particularly important for children from less privileged family environments, where the level of experience and knowledge is relatively low (Vecchiotti, 2001). This view also has a strong basis and influences educational decisions. Teachers when they find that there are age differences in the classroom between students and because they know the importance of the factor "age" in mathematical performance and general academic performance must make the necessary adaptations and adjustments. Especially in cases where the age differences are not very large and there are not very large differences in performance, the learning unit should be organized on the basis of objectives that will be a graded difficulty so that students can respond according to their level. The teacher should have identified what the student can learn and how they will learn it. Extensive use of the worksheet could be very effective. Each worksheet should include a specific learning objective as well as instructions for the process to be followed. The student should also be aware of the material at his disposal, to be able to easily access it as well as a variety of learning tools and resources. Especially in the subject of mathematics where the degree of abstraction is particularly high, the access and use of the material are of great importance. In cases where there are large age differences in a classroom and if these differences affect mathematical performance then the application of a different model of classroom organization should be considered. The class should be divided into groups and subgroups (each color differently). The purpose of segregation is to provide individualized teaching to students more effectively, to improve learning motivation, and to be able to function more effectively. Such an organization of the classroom enables the children to work in small heterogeneous groups, to help each other, and to consider each other's points of view. Heterogeneous groups create models of differentiation, which give students the opportunity to achieve different goals through alternative learning models.

The program could also include several regular breaks in which the students of the working groups can express and share their experiences, concerns and thoughts, especially the younger ones with the older students. To discuss and find solutions to the issues that arises.

Improving math skills should be a priority in kindergarten and the first grades of primary school. Until recently, however, early interventions related to mathematics have attracted much less interest than early intervention programs in reading (Fuchs, 2005; Gersten et al., 2005). However, the mathematical difficulties in kindergarten and primary school can be predicted to a large extent (Dowker, 2005), an event that is of particular importance.
CONCLUSIONS – PROPOSALS

The findings of the present study add to the growing body of research findings. The level of flowing intelligence and the age of the child are strongly related and seem to play an important role in the degree of mathematical competence. These factors can also predict the future mathematical level and therefore must be taken into account for the implementation of the necessary interventions at both educational and administrative level. We also believe that they can contribute to the planning and adoption of educational and policy interventions, measures and decisions in the direction of maximizing the educational result in the field of mathematical knowledge. At the educational level, the design and implementation of intervention programs must be approached with great care. Early learning is not just a "vaccination" that necessarily produces subsequent benefits. There are many factors to consider. Theories about the processes of creating and developing skills, the impact of skills in areas of mathematical knowledge, and the effective use of findings require constant research, continuous monitoring, data control, and possibly interventions and revisions. The results of the present study can be used in the development and implementation of programs that will improve the level of mental function in the age group of kindergarten students. Thus students will be able to have a higher level of mathematical competence which will result in better access to mathematical knowledge even in highly demanding areas such as the field of mental calculations. The implementation of intervention programs will result in the enhancement of interest and active participation along with an increased degree of enjoyment and satisfaction. Learning objectives are more easily achieved and the student can learn in a fun and engaging way.

At the decision-making level, educational policymakers such as the age of onset of kindergarten should rely on research data to maximize academic achievement and mathematical performance. Deciding on a child's enrollment age is certainly not an easy task. However, the fact that age plays a very important role in the mathematical performance and that the curriculum of the kindergarten is becoming more and more demanding must be taken seriously. Therefore, the educational community and educational policy-making institutions must have access to continuous and reliable information, the possibility of checking and an increased degree of flexibility in the application of new research data, the review of previous decisions and the continuous monitoring of measures, interventions and applicable regulations.
We believe that the present work provides evidence that the investigation of factors that affect the learning of mathematics in preschool children such as basic skills and cognitive functions is possible and necessary. With such an approach, a new dimension is given to the methodology of intervention in the field of mathematics. However, the role of factors that affect mathematical performance in other age groups besides kindergarten, such as first graders in elementary school, should be explored.

It should also be investigated which other factors affect and to what extent mathematical performance in different categories of students such as students with intellectual disabilities, students with learning disabilities or autism spectrum disorder (ASD), etc. It is also particularly important to investigate more the factors that can predict mathematical performance to enable early intervention programs.

It would be interesting to investigate the factors that influence or can predict performance in other academic areas (e.g. language courses, science courses, etc.) during students' pre-school education. In fact, the comparison of the role of these factors with each other would be of particular interest, allowing the design and implementation of more effective early intervention programs.

With the implementation of early intervention programs, the school innovates and effectively tackles stability and immobility, which often takes the form of stagnation with the main feature being the obsession with tradition. It is necessary for teachers to be informed about the new data from the research findings and to be encouraged to implement modern curricula and teaching methods. In this context, the area of pre-school education and basic education with options, modern and scientifically substantiated, must also move.

References


Analysis of the Mathematics Function Chapter in a Malaysian Foundation Level Textbook Adopted by a Public University

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Abstract: In order to enhance students learning environments, mathematics lecturers need to prepare suitable materials in their lessons. So improving the materials in textbooks based on learning theories is so important to enhance the ability of students in mathematics learning through problem solving activities. The purpose of this qualitative case study is to analysis the function chapter of a Malaysian foundation level textbook to prepare deeper understanding for lecturers about the quality of materials and to identify textbook elements which can improve to enhance the quality of learning. This study was conducted to investigate the quality of function chapter especially about problem solving in the Mathematics 1 textbook of a foundation center of a public university in Malaysia in 2019. The method of this research is content analysis. In this current study, three theories namely Bloom’s taxonomy, behaviorism and constructivism were used to analyses the textbook’s materials. The findings represented that exercise solving based on the behaviorism theory highlights this textbook instead of problem solving based on the constructivism theory. Meanwhile, the textbook’s materials mostly are related to the first two levels of Bloom’s taxonomy. Finally, some errors in the textbook are addressed in order to lecturers improve them for the next editing.

Keywords: Mathematics textbook, Mathematics function, Mathematics problem, Mathematics exercise, Higher order thinking
1. Introduction

The materials considered in the textbook as a primary instrument in teaching and learning mathematics are so important to enhance the students’ ability in mathematics problem solving (Berisha et al., 2013). Doing suitable activities by students help them to have better performance in the classes and improve their abilities in mathematics problem solving. Brandstrom (2005) explained that the textbook has a very central task in the classroom for educators and learners in mathematics teaching and learning process. The mathematics function concept is a central and practical but difficult topic in secondary school curricula (Akkus et al., 2008; Ponce, 2007). For instance, “the topics inverse function and composite function is more conceptual and challenging among educators to transfer to students” (Oehrtman, Carlson, & Thompson, 2008, p.39). Mathematics functions apply in human life to modeling the real-world problems (Michelsen, 2006). Therefore, students should learn mathematics function conceptually as a practical topic in human life.

The quality of mathematics materials in the textbooks plays an important role in teaching method among lecturers. If the materials in textbooks contain suitable problems and activities, lecturers need to improve their content knowledge and pedagogical content knowledge to use problem solving approaches in their classes. Otherwise, lecturers, by using low quality textbook through traditional method, cannot improve the students’ abilities in problem solving. Educators utilizing traditional method of teaching mostly emphasize on the importance of lecture and mathematics exercise solving among students (Khalid, 2017; Mon et al., 2016). In fact, in this lecturer-centered approach of teaching, many of students only memorize the theorems, formulas and methods of solutions and apply them in mathematics exams or homework problems (Gholami et al., 2019). In mathematics education, learning theories are used to explain how students learn the new concepts. Therefore, the learning theories help educators and mathematics experts to understand the complex process of learning among students. In educational studies, researchers discuss two main perspectives in learning theories namely Behaviourism and Constructivism. Traditional method in mathematics teaching is supported by the behaviorism learning theory in education. Behaviorist refers to the learning theories emphasizing changing behavior which is resulted from learners’ associations of stimulus-response (Ormord, 1995). According to this theory, learning among learners is a change in their behavior because of their experience (Ormord, 1995). The problem solving approach is based on the constructivism learning theory. Constructivism is the theory that
students construct their own understanding of the mathematics concepts through experiencing problem solving activities and reflecting on those experiences (Simmons, 1999). In fact, mathematics problem solving helps learners develop a wide domain of complex mathematics structures and obtain the ability of modeling variety of real-life problems (Tarmizi & Bayat, 2012).

In behaviorism learning theory, educators does not teach critical thinking, rather it excludes any form of cognition (Von Glasersfeld, 2008). Behaviorists think that mathematics knowledge and materials transfer from one person to another by means of reinforcements and conditioning. This kind of mathematics learning involves rote learning, repetition and external rewards to elicit behavior. This educational theory prepares an incomplete way to teaching according to memorization method. Some appropriate behaviorist strategies need to be performed in order to encourage and motivate participants to learn the basic knowledge of mathematics. For instance, when lecturers teach the concept of composite function, it is appropriate to consider some mathematics exercises as students’ activity to help them learn the definition and concept of composite function to engage students with suitable problems based on their abilities.

Constructivist methods in mathematics teaching are student-centered and provide a suitable environment for students to learn the concept of mathematics deeply in a group. In fact, this learning theory emphasizes mathematics problem solving among learners individually and teamwork. According to constructivists, successful learning is the skill of the learner to explain procedures that would best interpret the environment (Ogwel, 2004). Bettencourt (1993) described constructivism as a theory that “involves a conception of the knower, a conception of the known, and a conception of the relation of knower-known” (p.39). Ogwel (2004) explained that “more still, to the constructivists, emphasis should be on the process of learning and not the product of learning” (p.2).

2. Theoretical Framework

Problem solving is so important in the process of mathematics learning. The teaching methods focusing on problem-solving has been a hot topic in the field of mathematics education among researchers and educators during last two decades (Hu et al., 2018). Therefore, including the suitable mathematics problems in the textbooks prepare vast opportunity for students to learn mathematical concepts deeply. According to the National Council of Teachers of Mathematics (NCTM) (2000), if students engage with a challenging task for the first time then this task is known...
as mathematics problem. On the other hand, if students follow some steps to solve a routine task then this task is called mathematics exercise. Therefore, problem solving points to engaging in a mathematics question that learners have not learned how to solve it before. Based on the study by Gholami et al. (2019) “the distinction between what is considered a problem and an exercise depends on many factors, including the grade level, mathematics competence, learning materials, the way it was taught, and the time given to complete the task” (p. 292). Asami-Johansson (2015) explained that open-ended problems and the level of problems depend on the students’ mathematical ability. For instance, the following mathematics problem, after discussion in class, becomes a mathematics exercise.

Problem 1: If \( \tan(x + y) = \frac{2}{3} \) and \( \tan(x - y) = \frac{1}{4} \) then find the value of \( A = \tan 2y - \cot 2x \).

If lecturers consider a slightly change in this mathematics exercise, students will engage with another mathematics problem such as:

Problem 2: If \( \tan(x + y) = \frac{2}{3} \) and \( \tan(x - y) = \frac{1}{4} \) then find the value of \( B = \tan(3y + x) + 2 \tan(4x) \).

Furthermore, in this research, every mathematics problem related to the student’s everyday life and other subjects such as physics, chemistry and biology are considered as practical problem. For example, the following task is a practical problem.

A farmer is growing winter wheat. The amount of wheat he will get per hectare depends on, among other things, the amount of nitrogen fertilizer that he uses. For his particular farm, the amount of wheat depends on the nitrogen in the following way:

\[
Y = 7000 + 32N - 0.1N^2
\]

Where \( Y \) the amount of wheat is produced, in kg per hectare, and \( N \) is the amount of nitrogen added, in kg per hectare.

i. How much wheat would he have if he uses 200 kg of nitrogen per hectare?

ii. Could you have better suggestion for him to use the amount of nitrogen to produce more amount of wheat?
Polya (1945) suggested four phases for mathematics problem solving, namely, understanding the problem, planning a strategy, performing the plan, and confirming the answer. Since 1945 a lot of models for problem solving with different steps and phases have been introduced by educators and researchers but all models have described that students should understand the problem, choose a strategy, solve the problem and confirm the answer. It is important that mathematics textbook should be able to encourage and engage students with suitable problems according to their abilities and skills. The discovery of the use of mathematics problems and encouraging learners to describe the techniques and strategies they engage when solving problems are more pedagogically challenge among mathematics educators (Johnson & Cupitt, 2004; McDonald, 2009). So educators can improve the students’ abilities and skills in problem solving and higher order thinking by engaging them with appropriate mathematics activities in the textbooks.

Educators use Bloom’s taxonomy in mathematics assessments in order to ensure that learners assess on a variety of skills in problem solving. The revised Bloom’s taxonomy (2001) describes the levels of learning in six categories, namely, remembering, understanding, applying, analyzing, evaluating, and creating (Anderson et al., 2001). There is no higher order thinking skills without lower order thinking skills. Therefore, there is a strong relationship between lower order thinking and higher order thinking (Mitana et al., 2018). When students engage with difficult mathematics problems, they need to use some definitions, theorems and methods that they have memorized or understood before. For instance, in the problem “Find the value of $A = \sin 22.5^\circ + \cos 22.5^\circ$.” students can solve this problem in variety solutions methods such as

$$A = \sin 22.5^\circ + \cos 22.5^\circ \Rightarrow A^2 = (\sin 22.5^\circ + \cos 22.5^\circ)^2$$

$$A^2 = \sin^2 22.5^\circ + \cos^2 22.5^\circ + 2 \sin 22.5^\circ \cos 22.5^\circ$$

$$A^2 = 1 + \sin 45^\circ = 1 + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2} \Rightarrow A = \sqrt{\frac{2 + \sqrt{2}}{2}}.$$  

Although this solution shows the students’ higher order thinking skills, some facts related to the lower order thinking skills are used in this solution such as $(a + b)^2 = a^2 + b^2 + 2ab$, $\sin 2\theta = 2 \sin \theta \cos \theta$ and $\sin 45 = \frac{\sqrt{2}}{2}$. In this study, based on the definition of lower and higher order thinking skills by Malaysian Ministry of Education (Ministry of Education, 2014), the first two levels of revised Blooms’ taxonomy, remembering and understanding are considered as the
skills of lower order thinking and four levels applying, analyzing, evaluating and creating are considered as the skills of higher order thinking. Berry, Maull, Johnson and Monaghan (1999) introduced the routine mathematics questions as lower order thinking skills that students can solve them easily using some steps and procedures. Non-routine questions need the application of mathematics materials such as definitions, theorems and methods in a new situation and critical thinking to find creative solutions. They further added one question may be considered routine for a student but non-routine for another. In other words, the categorization of mathematics tasks based on revised Blooms taxonomy depends on the students’ abilities in problem solving. Therefore, mathematics education experts can do the classification of mathematics tasks in different levels of the revised Blooms taxonomy according to the students’ problem-solving skills. For instance, Dartington (2013) categorized the following pre-university level questions in the levels of higher order thinking because in these questions, students engage with challenging mathematics problems and they require to think critically.

Example 1: A curve’s equation is \( y = f(x) \), where \( f(x) = \frac{3x+1}{(x+2)(x-3)} \). Express this in partial fractions.

Example 2: The matrix \( A \) is \( A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \). Prove by induction that, for \( n \geq 1 \),

\[
A^n = \begin{pmatrix} 3^n & \frac{1}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}.
\]

The importance of using suitable materials in the textbooks in the process of mathematics teaching is supported by many educational theories and hence, preparing mathematics material in the course books based on the learning theories enable acquisition of problem solving skills among students such as making predictions and judgments, intuitive thinking, abstract thinking and extracting mathematical formulas to model the real world problems (Koparan, 2017). It seems the analysis of mathematical materials based on the theories of Bloom’s taxonomy, behaviorism and constructivism not only provide golden opportunities for the lecturers to understand the weaknesses of the materials according to the different levels of the Bloom’s taxonomy but also help them to enhance the skills of higher order thinking among students. The purpose of this current study is to analysis the mathematical materials in the function chapter of the Mathematics 1 textbook as respect to the problem solving and higher order thinking to prepare deeper
understanding for lecturers about the quality of materials and to identify textbook elements to enhance the quality of learning. As scope of this research, the following research question was aimed to be answered:

Research Question: Is the mathematical materials in the function chapter of the Mathematics 1 textbook, appropriate in terms of problem solving and higher order thinking?

3. Methodology

This current study was part of a larger research study that was conducted in a foundation center in a public university of Malaysia during the academic year 2018-2019. Foundation program is a kind of Malaysian pre-university programs that run by selected universities. Several other universities also offer foundation programs in one year (two semesters) and the learners are almost similar in their qualifying entrance grades. Students pursuing university level are selected based on their performance in foundation level. So, students should engage with suitable mathematics materials to improve their abilities in problem solving and prepare them for better performance in mathematics courses at the university level. In foundation center, students used two mathematics textbooks, namely, Mathematics 1 and Mathematics 2 in semesters one and two, respectively. It is worth mentioning that these textbooks are taught by lecturers during one year (two semesters). The principal of the Foundation Center and the Head of Mathematics Unit signed the permission letter. The Head of Mathematics Unit explained that these textbooks are designed by all mathematics lecturers in this center (each lecturer designed one chapter) and these textbooks contain different chapters related to the algebra, calculus, trigonometry, geometry, probability and statistics.

The method of this qualitative case study is content analysis of the Mathematics 1 textbook (version 2018) of a foundation center in a Malaysian public university. In the Mathematics 1 textbook, there are fifteen chapters that the researchers chose the function chapter randomly. This study aims to analyses the materials of the function chapter in order to find the appropriateness of them as respect to the problem solving and higher order thinking. The five topics related to the function chapter are shown in Table 1. Pedagogical approaches to these topics consist of following three theories: revised Bloom’s taxonomy, behaviorism and constructivism. Furthermore, this
paper tried to identify textbook elements which can be improved in order to enhance the quality of teaching and learning in mathematics.

Table 1: The Topic of Lessons

<table>
<thead>
<tr>
<th>Number</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Relation and function concepts</td>
</tr>
<tr>
<td>2</td>
<td>Domain and range of the functions and algebraic combination</td>
</tr>
<tr>
<td>3</td>
<td>Composite function, inverse function, odd and even functions</td>
</tr>
<tr>
<td>4</td>
<td>Trigonometric functions</td>
</tr>
<tr>
<td>5</td>
<td>Exponential and logarithmic functions</td>
</tr>
</tbody>
</table>

In this study, the researchers, the Head of Mathematics Unit and a lecturer were divided all the tasks of the Mathematics 1 textbook into two categories, mathematics exercises and mathematics problems according to the definitions of mathematics problem and mathematics exercise (NCTM, 2000). Mathematics exercises are regarded as questions solved using similar tasks, while mathematical problems are considered as applying these tasks to more challenging problems. Also, the mathematics problems were categorized based on the revised Bloom’s taxonomy. These categorizations and the analysis of textbook materials were improved and confirmed by three professors in the faculty of mathematics at the same university through content review. Meanwhile, some factors about errors in typing, definitions and students misunderstanding are explained and addressed that deserve full consideration by lecturers in the next edition. Meanwhile, this textbook was designed in 2018 and lecturers taught it for the first time. The new textbook design allowed for researchers to study its strengths and weaknesses to improve upon in subsequent editions.

4. Findings

The findings of this study about the functions in the Mathematics 1 textbook are discussed in two parts, namely, problem solving and higher order thinking, and critique of the content.

4.1. Problem Solving and Higher Order Thinking

The analysis of materials related to these five topics in the Mathematics 1 textbook were insubstantial related to the mathematics problems. There are a few mathematics problems in the
textbook. Meanwhile, there are not any practical problems in this chapter. In fact, the textbook focuses on solving mathematics exercises which is related to the behaviorist learning theory. This method encourages students to memorize some methods, formulas, theorems and shortcuts in order to apply them in some mathematics exercises using the lecturers’ methods and steps. These lessons cannot improve their abilities in problem solving based on constructivism learning theory. For example, in the subtopic of composite function there are 18 similar routine questions on page 157 such as “find $fog$ for the functions $f(x) = \sqrt[3]{x}$ and $g(x) = x^2 - x - 6$” that only the function rule of tasks is different in these questions. The questions “solve the trigonometric equation $4\sec^2\theta = 3 \tan \theta + 5$” on page 159 and “prove that for any positive, real number $x$ we have $\ln\left(\frac{1}{x}\right) = -\ln x$” are examples of mathematics problem because they are challenging task for students. The result of analyzing the textbook materials is represented in Table 2.

Table 2: The Number of Mathematics Problems in the Textbook

<table>
<thead>
<tr>
<th>No.</th>
<th>Topic</th>
<th>Exercise</th>
<th>Problem</th>
<th>Practical Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Relation and function concepts</td>
<td>16</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>Domain and range of the functions and algebraic combination</td>
<td>19</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>Composite function, inverse function, odd and even functions</td>
<td>26</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>Trigonometric functions</td>
<td>10</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>Exponential and logarithmic functions</td>
<td>18</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td><strong>Total</strong></td>
<td><strong>89</strong></td>
<td><strong>6</strong></td>
<td><strong>0</strong></td>
</tr>
</tbody>
</table>

Regarding Table 2, in the case of these five topics, about 6.3% of tasks are mathematics problems and 93.7% of tasks are mathematics exercises. Meanwhile, there are no practical problems in each lesson to encourage students in learning mathematics by seeing some application of mathematics in the real world.

The mathematical tasks categorized by lecturers and three professors from the mathematics faculty confirmed them. For instance, the task “find the range of the function $h(x) = \frac{1}{\sqrt{4-x^2}}$” were
categorized into applying level of revised Bloom’s taxonomy because this task is not a routine question and students should find its inverse to consider the domain of invers function as the range of the function $h$. Table 3 shows the categories of tasks according to the revised Bloom’s taxonomy for all topics.

Table 3: The categorization of Tasks Based on the Revised Bloom’s Taxonomy

<table>
<thead>
<tr>
<th>Topic</th>
<th>Task</th>
<th>Remembering</th>
<th>Understanding</th>
<th>Applying</th>
<th>Analyzing</th>
<th>Evaluating</th>
<th>Creating</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17</td>
<td>5</td>
<td>11</td>
<td>1</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>2</td>
<td>20</td>
<td>7</td>
<td>12</td>
<td>1</td>
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</tr>
<tr>
<td>3</td>
<td>28</td>
<td>14</td>
<td>13</td>
<td>1</td>
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<tr>
<td>4</td>
<td>11</td>
<td>4</td>
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<td>5</td>
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<td>6</td>
<td>12</td>
<td>1</td>
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<td>0</td>
<td>0</td>
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<td>53</td>
<td>4</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

With respect to Table 3, about 94% of tasks are related to the lower order thinking and 6% of tasks are about higher order thinking. Therefore, the materials in the textbook are not appropriate for foundation level students to improve their higher order thinking skills.

4.2. Review of the Textbook’s Materials

On page 143 of the textbook, a relation is defined as “the association or relationship between two sets of information or objects which is called a relation and every set contains some ordered pairs which is considered a relation”. It seems that this definition is not appropriate for foundation level textbook. The professors who reviewed the content of the textbook suggested that it is better to define the relation by using Cartesian product $(A \times B = \{(x, y) | x \in A, y \in B\})$ because students can learn the concept of the function logically. On page 144, a function is defined as “a relation that assigns each input of $x$-values of the domain to exactly one output of $y$-values of the range.”. For this case, considering a simple example like, “the height of an airplane in different time shows a function” in the textbook is useful to prepare the students mind for understanding the concept of the function conceptually. Because it is not possible that at the same time an airplane has two different heights, but it has the same height in two or more different times. Lecturers should consider an activity about this concept to provide some examples of function in daily life.
In the foundation level, the mathematics concepts should be transferred through logical and accurate materials. On page 148, we see the following definition “rational function is defined by $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials”. In this definition, we need to consider $q(x) \neq 0$. Another definition provided is “root function is defined by $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$, where $n$ is a positive integer, and the domain of the root function is the set of real numbers if $n$ is odd, and the set of all positive real numbers if $n$ is even”. Two points are important and should be considered in this definition. Firstly, in the function $f(x) = \sqrt[n]{x}$ the value of $n$ cannot be one and secondly, the domain of this function is non-negative real numbers.

Typographical errors create some challenges for students. In this chapter, there are some typographical errors. On page 158, there are two trigonometric formulas that are represented incorrectly $\sin x \cos y = \frac{1}{2}(\sin(x + y) - \sin(x - y))$ and $\cos x \cos y = \frac{1}{2}(\cos(x + y) - \cos(x - y))$. Using these incorrect formulas confuses students. These formulas should be corrected as follows:

$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y))$ and $\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y))$.

Some mistakes in the mathematics concepts may lead to misunderstanding. On page 159, there is a sentence “for any $x, y \in (-\infty, +\infty)$, $\ln xy = \ln x + \ln y$” where the domain of the function $f(x) = \ln x$ is $(0, +\infty)$. In this case both variables $x$ and $y$ should be positive, real numbers.

Besides, there are some scientific problems such as the following definitions of even and odd functions: On page 156, the definition of even and odd functions are ambiguous for students and lead to misunderstanding. An even function is defined as follows:

“A function $f$ is said to be even if and only if $f(-x) = f(x)$ for all $x$.”

Also, an odd function is defined as:

“A function $f$ is said to be odd if and only if $f(-x) = -f(x)$ for all $x$.”

For example, students apply the above definitions for the question “the function $f(x) = x^2$ with $-3 \leq x \leq 2$ is even or not?” as this is an even function because this function satisfies the condition $f(-x) = (-x)^2 = x^2 = f(x)$. But this function is not even because the graph of this
function is not symmetric with respect to the \(y\)-axis. For another example, some students by using the textbook definitions explain that the function \(h(x) = \sqrt{2x} + \sqrt{-2x}\) is even because \(h(-x) = \sqrt{-2x} + \sqrt{2x} = \sqrt{2x} + \sqrt{-2x} = h(x)\). But for this function \(D_h = \{0\}\) therefore, \(h = \{(0, 0)\}\) it means that the function \(h(x) = \sqrt{2x} + \sqrt{-2x}\) is both even and odd. Thus these definitions should be changed as follows:

A function \(g\) is called an even function if the following two conditions are met.

a. Domain \(g\) is symmetric with respect to the point \((0, 0)\)

b. \(\forall x \in D_g, g(-x) = g(x)\)

A function \(h\) is called an odd function if the following two conditions are met.

a. Domain \(h\) is symmetric with respect to the point \((0, 0)\)

b. \(\forall x \in D_h, h(-x) = -h(x)\)

There are a limited number of mathematics problems in this section. Considering suitable problems in the textbook such as “how many functions both even and odd can we find?” can improve students’ ability with problem solving. Lecturers can help students to solve this challenging problem as follows:

Since \(f\) is even \(f(-x) = f(x)\) also, \(f\) is odd \(f(-x) = -f(x)\) therefore,

\[
f(x) = -f(x) \Rightarrow 2f(x) = 0 \Rightarrow f(x) = 0.
\]

Based on this argument, there is only one function that is both even and odd. If we consider different domains for the function \(f(x) = 0\) then we can find many different both even and odd functions. For example:

\[
f = \{(-4, 0), (-2, 0), (0, 0), (2, 0), (4, 0)\} \quad \text{or} \quad g(x) = \begin{cases} 
[x] & \text{if } 0 \leq x < 1 \\
1 + [x] & \text{if } -1 < x < 0.
\end{cases}
\]

On page 156, there is a mathematics problem related to even and odd functions as follows:

Problem: Show that
a. The sum and difference between even functions are even.
b. The sum and difference between odd functions are odd.
c. The sum and difference between even and odd functions are neither even nor odd.
d. The product between even functions is even.
e. The product between odd functions is even.
f. The product between even and odd functions is odd.

Although this is a good problem, all parts rejected considering a counter example $h(x) = 0$ and $k(x) = 0$. In the other words, all parts of this problem are incorrect. For instance, for part (a) if we consider $h(x) = 0$ and $k(x) = 0$ then $(h + k)(x) = h(x) + k(x) = 0 + 0 = 0$ thus the sum of two even functions is odd. Therefore, this problem should be corrected as follows:

Problem: Prove that

a. The sum and difference between non-zero even functions are even.
b. The sum and difference between non-zero odd functions are odd.
c. The sum and difference between non-zero even and non-zero odd functions are neither even nor odd.
d. The product between non-zero even functions is even.
e. The product between non-zero odd functions is even.
f. The product between non-zero even and non-zero odd functions are odd.

There are many similar mathematics exercises in each topic that consume a lot of time without improving the students’ learning. For example, in topic 3, there are 18 exercises related to composite functions like the following exercise:

“If $f(x) = 1 - x$ and $g(x) = \frac{1}{x^2 + 1}$ find the function $fog$.”

For this example, lecturers can consider some problems to improve students’ abilities and skills in problem solving such as:

Problem: If $f = \{(1,2), (3,5), (5,8), (4, -1)\}$ and $g = \{(2, -3), (3,1), (5,7), (-2,4)\}$ then find the function $fog + gof$.

Problem: If $f(x) = 2x - 5$ and $(gof)(x) = \frac{x-2}{x-4}$ find the function $g(x)$.
In the textbook, the domain of composite function discussed based on composite function rule that this method sometimes makes a misunderstanding for students. Thus, the logical definition $D_{fog} = \{ x \in D_g | g(x) \in D_f \}$ is necessary to improve the ability of students in problem solving. For example, in the problem “If $f(x) = \frac{x+1}{x-1}$ and $g(x) = \frac{1}{x-2}$ then find the domain of the function $fog$” according to the method of this textbook, students first find the function $fog(x)$ as:

$$fog(x) = f(g(x)) = f\left(\frac{1}{x-2}\right) = \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \frac{\frac{1+x-2}{x-2}}{\frac{1}{x-2}} = \frac{x-1}{x-1}$$

Secondly, the domain of the function $fog(x) = \frac{x-1}{3-x}$ is $(-\infty, 3) \cup (3, +\infty)$. However, according to the logical definition of the domain of composite function, we have:

$$D_{fog} = \{ x \in D_g | g(x) \in D_f \} = \{ x \in (-\infty, 2) \cup (2, +\infty) | \frac{1}{x-2} \neq 1 \} = \{ x \neq 2 | x - 2 \neq 1 \} = (-\infty, 2) \cup (2, 3) \cup (3, +\infty)$$

Although, there are some weaknesses in this chapter there are some strengths such as covering different kind of functions that are applicable to other majors. Also, the authors organized and linked the topics appropriately. For instance, in topic 2 of the textbook, the authors considered a challenging problem related to the range of the function $f(x) = \frac{x+1}{x-2}$ that usually students cannot solve it with their prerequisite mathematical knowledge. According to the ideas of lecturers in this foundation center, most of students cannot find the range of this function despite spending a lot of time. When lecturers teach the concept of inverse function in the topic 3 and explain that for two functions $f$ and $f^{-1}$ we have $D_f = R_{f^{-1}}$ and $R_f = D_{f^{-1}}$ then students learn that one of the important applications of inverse function is to find the range of some functions. Therefore, they can find the inverse of $f(x) = \frac{x+1}{x-2}$ as $y = \frac{x+1}{x-2} \Rightarrow x = \frac{y+1}{y-2} \Rightarrow y = \frac{2x+1}{x-1} \Rightarrow f^{-1}(x) = \frac{2x+1}{x-1}$ and they easily see that $R_f = D_{f^{-1}} = (-\infty, 1) \cup (1, +\infty)$. For another example, two problems 7 and 8, on page 157 are about the composite function $fogoh$ it is a suitable example for students to generalize the definition of composite function. The exercise “identify the possible functions $f(x)$ and $g(x)$, given that $(gof)(x) = \ln (2x + 2)$” is an appropriate task for students to find different functions such as $f(x) = 2x + 2$ and $g(x) = \ln x$ or $f(x) = 2x$ and $g(x) = \ln(x + 2)$ or $f(x) = x + 1$ and $g(x) = \ln 2x$, recognizing that these functions are not unique.
5. Discussion and Conclusion

The results of this study confirmed that in this foundation center, the function chapter in the Mathematics 1 textbook emphasizes solving mathematics exercise based on behaviourism learning theory. A few of the tasks (6.3%) related to solving mathematics problem according to constructivism theory. Hence, problem solving is poor throughout the textbook. Furthermore, about 6% of tasks related to the higher order thinking skills. In Mathematics 1 textbook, each chapter is written individually by one of the lecturers. It seems that lecturers through collaborative work and using different educational theories can significantly increase the quality of this textbook. Thus, lecturers can collaboratively discuss the textbook material and decide how to improve their teaching. It is so important that mathematics lecturers need to have a strong foundation of learning theories and frameworks while planning to teach the materials in the textbook. They should be required to improve their knowledge about the learning theories such as the Bloom’s taxonomy, behaviorism and constructivism in order to provide suitable materials in the textbook. For example, in Mathematics 1 textbook, there are 18 mathematics exercises about composite functions that emphasize drill-and-practice. There are similar examples based on behaviorism theory and the first two levels of the revised Bloom’s taxonomy, remembering and understanding. But lecturers with suitable knowledge about learning theories and context can collaboratively improve the topic on composite functions. Based on the context, they should find how many mathematics exercises based on behaviorism theory about two levels remembering and understanding of revised Bloom’s taxonomy and how many mathematics problems based on constructivism theory about the higher order thinking skills or four levels applying, analysing, evaluating and creating of revised Bloom’s taxonomy should be considered in this lesson. Meanwhile, considering some practical problems could help students have better attitudes toward mathematics.

In this foundation center, the majority of mathematics lecturers taught precisely the same textbook materials, and the quality of the textbook about the mathematics function was deficient. Mathematics function is one of the essential topics used in many mathematics courses at the university level. Students in a foundation-level course need to have proper knowledge about the functions. So, they need to engage with practical problems in the textbook to learn the mathematics concepts meaningfully and experience its beauty. For example, suppose students learn even and odd functions conceptually. In that case, they can apply problem techniques and the properties of these functions to other related mathematics topics such as range of the functions and integration.
The Mathematics 1 textbook designed newly and taught for the first time in this foundation center thus it is natural to find some conceptual and typographical errors in different chapters including the function chapter. Therefore, this article could help lecturers improve the quality of the textbook according to the different learning theories. In fact, considering less lectures and more problem solving activities in regarding different mathematical topics through using constructivist learning theory has an important role in mathematics learning among students. Although this study focused on the function chapter, the results of this research still can be generalized for other chapters as well. For instance, the situation of mathematics problem solving can be improved throughout all chapters by lecturers in the new edition of textbook. For this case, collaboratively work among lecturers is so essential to share their knowledge, skills and experiences in order to produce a suitable textbook for foundation program students.

References


Understanding Proof Practices of pre-Service Mathematics Teachers in Geometry

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Abstract: In this work, we show the results of a research with pre-service mathematics secondary teachers about their Van Hiele level regarding the proof in Geometry. We observe three different profiles whose characteristics are described. These descriptions allow us to foresee certain differences when carrying out proof teaching in secondary school. The presence of a profile with a lower level than that assumed for some high school students stands out. The other two profiles show differences regarding the presence of some advanced proof strategies.

Keywords: proof, Van Hiele levels, pre-service teachers

INTRODUCTION AND OBJECTIVES

The teaching of proof has been a concern in the context of secondary-school teacher training due to the educational knowledge shown by pre-service teachers (Arnal-Bailera & Oller-Marcén, 2017; Dos Santos & Ortega, 2013; Makowski, 2020). In order to help pre-service teachers to develop a solid knowledge of proof, it is important that mathematics teacher educators become aware of how pre-service teachers understand proving processes (Stylianides, Stylianides & Philippou, 2007).

Several studies have also addressed the level of development of pre-service mathematics teachers’(PSMT) geometric thinking following the Van Hiele model (Lee, 2015; Mayberry, 1983; Pandiscio & Knight, 2011; Wang & Kinzel, 2014). According to Güler (2016), lack of prior knowledge, proof methods understanding, students’ memorization of proofs instead of questioning them and biases against proof are the main difficulties for PSMTs in this matter. There are other studies explaining particular difficulties during proving processes such as the...
understanding of the meaning associated with the inductive step in a proof by induction or the logical equivalence of two affirmations (Stylianides et al., 2007).

In this study, we aim to characterize the various ways that pre-service teachers proof in Geometry, in particular their use of graphics (Mesquita, 1998), discourse (Duval, 1995) and proof schemes (Harel & Sowder, 1998). This detailed description of the proof practices would help mathematics teacher educators identify and understand the different levels shown by pre-service teachers and help them to progress from one level to the next. Moreover, our study tries to use the Van Hiele model lens to compare the levels shown by pre-service teachers with the levels required by pre-college curricula to secondary school students and, implicitly, to their teachers. In the National Council of Teachers of Mathematics [NCTM], (2000, p. 310) the importance of proof in mathematics education is clearly stated: “Students should see the power of deductive proof in establishing the validity of general results from given conditions.”

We conducted a study with twenty-five students enrolled in a master’s degree in secondary school teacher training at the University of Zaragoza with various profiles relating to access, prior training and age. We administered them a questionnaire with open-ended questions designed to assess their Van Hiele level of geometric reasoning. In this article we will limit ourselves to analyzing the issues relating to the proof, even though we realize that undertaking the teaching of a proof also requires knowing details about the students, curriculum, and so on.

The research question we wanted to answer was: Do future secondary school teachers have the Van Hiele level needed to undertake the teaching of proof that current secondary curricula require? To answer this question, we had two research objectives:

- To identify different profiles of pre-service secondary mathematics teachers according to the geometric reasoning shown in their proof practices.
- To describe pre-service secondary mathematics teachers proof practices in Geometry.

THEORETICAL FRAMEWORK

Van Hiele model

In the 1950s, Van Hiele (1957) and Van Hiele-Geldof (1957) elaborated a model that describes the development of geometric thinking. This model establishes that there are five levels of geometric reasoning (Hoffer, 1983; Van Hiele, 1957, 1986). As these levels are also sequential and hierarchical, students pass through them in a specific order without omitting any of them throughout the geometry learning process. The strictly hierarchical nature of the levels has been
questioned in the later decades. In (Gutierrez & Jaime, 1998) the authors describe that Van Hiele levels are local, meaning that people can exhibit different level of reasoning at different subtopics of geometry. Furthermore, a student can also reason at various levels on different tasks (Burguer & Saughnessy, 1986; Clements & Battista, 1992).

In this study, we will focus on the three intermediate levels: level 2 (Analysis); level 3 (Informal deduction); and level 4 (Formal deduction).

In the context of this study, focusing on proof, level 2 is characterized by proofs limited to verifying whether a certain property is fulfilled in a few particular cases. In level 3, the properties can be verified in some examples, although with informal explanations based on mathematical properties. Lastly, formal mathematical proofs are conducted in level 4 (Jaime & Gutiérrez, 1994).

Producing questionnaires that correctly measure the Van Hiele levels of geometric reasoning has been a well-studied subject for many years. Usiskin (1982) prepared a multiple-choice test comprising twenty-five questions (five for measuring each level). Burger and Shaughnessy (1986) designed an interview questionnaire comprising eight activities. These authors also described level indicators they use to place each student’s answer in one level or another.

Allocating Van Hiele levels involves some difficulties. These include where to place students that show signs of being between two consecutive levels. To solve this difficulty, Gutiérrez et al. (1991) propose an alternative form of assessing Van Hiele levels by describing a way of not only obtaining the level the students are at, but also the extent to which they have acquired this level. The authors describe eight answer types (0-7) in their study. These indicate varying levels of acquisition within the same level. Consequently, on evaluating an answer, we can place it within one of these types to allocate a numeric level of acquisition in accordance with Table 1:

<table>
<thead>
<tr>
<th>Type</th>
<th>0</th>
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<td>20</td>
<td>25</td>
<td>50</td>
<td>75</td>
<td>80</td>
<td>10 0</td>
</tr>
</tbody>
</table>

Table 1. Degrees of acquisition

Authors such as De Villiers (1987), however, study Van Hiele levels by analyzing several processes or components. Jaime and Gutiérrez (1994) describe the key processes they observe for Van Hiele levels: Identification (establishing which family a certain geometrical object
belongs to); definition (use and formulation of definitions of geometrical objects); classification (placing different geometrical objects into different families); and proof (statement tests). A test with eight items is presented in (Gutierrez & Jaime, 1995); it can assess each student’s first four Van Hiele levels and their degree of acquisition of each level. The key assessed processes for each of the issues are also described. This test has a higher reliability than multiple-choice questionnaires and it can also be administered to larger samples than interview questions. That is why we will use this test as a tool in this study.

Pandiscio and Knight (2010) analyzed the Van Hiele levels of pre-service secondary mathematics teachers finding that they did not attain level 4. These researchers stated that level 4 should be fully acquired by pre-service teachers since secondary school students should be guided to complete the acquisition of level 3 and start the acquisition of level 4. These ideas are congruent with previous studies (Mayberry, 1983) where she stated that “The response patterns suggest that these students were not at the proper level to understand formal geometry […] any high school geometry textbook will show […] that Level III should be developed” (p.68). Note that Mayberry referred level 4 as level III. Gutierrez & Jaime (1995) analyzed different profiles of secondary students attending to their van Hiele levels noticing that 12.9% of the students in grade 12 are in transition between levels 3 and 4. Due to all these previous reasons we agree with Pandiscio and Knight (2010) that PSMTs should present certain acquisition of van Hiele level 4 in order to promote the transition between levels 3 and 4 of their students.

**Proof**

Harel and Sowder (1998) described a student’s proof scheme as “what constitutes ascertaining and persuading for that person” (p. 244) and classifies proof schemes using three non-independent categories: externally based, empirical, and analytical. External conviction proof schemes are based on outside sources that influence students’ conceptions of proofs including authoritarian (the outside source is a teacher, a textbook …), ritual (the outside source is the traditional format of the proofs) and symbolic schemes (the outside source is the blind faith in the use of symbols independently of its meaning in the situation under consideration). The second category, empirical proof schemes, includes the proving or disproving of conjectures utilizing visual perceptions or examples-based proofs. The third category, analytical schemes, includes transformational (the reasoning is oriented toward settling the conjecture in general) and axiomatic proofs (the reasoning is organized so that any result is a logical consequence of the previous ones).
In their research, Harel and Sowder (1998) found that the depth of the pre-service teachers’ mathematical knowledge influenced the primary proof schemes utilized. In particular, middle school pre-service mathematics teachers used external conviction as their primary proof scheme, while teachers following a dual program (middle/secondary) showed empirical proof schemes and secondary pre-service mathematics teachers used a variety of schemes (Sears, 2019). Makowski (2020) pointed out that middle school PSMTs proofs rely mainly on inductive justifications as well as Demiray and Işıkşal (2017) who found that those PSMTs think that numerical values and examples were more convincing than mathematical proofs while Weber (2010) stated that most of the mathematics majors did not accept empirical arguments as proof after receiving appropriate training. Other studies (Uğurel et al., 2015) focused on the errors showed in most of the pre-service teachers’ proofs. Some of these errors were: failing to know where to getting started on a proof, showing prejudices towards construction of proof, feeling uncomfortable when constructing a proof, showing some lack of knowledge related to mathematical language and notation, method, concept and communication related problems in the proving process, and lack of content and strategy knowledge regarding the proof. This group of researchers stated that the main problems were related more to the understanding of the proving process than to the knowledge required for the proof itself.

De Villiers (1993), describes five proof functions: Verification/Conviction (establishing the truth of a statement); Explanation (proving why the demonstrated statement is true); Systematization (organizing several results into a global system); Discovery (making it possible to arrive at new results arising from the proof); and Communication (conveying mathematical knowledge). When proof is used in the classroom, the functions shown are verification and explanation (Crespo & Ponteville, 2005). Some of these functions, but not all, appear in the questionnaire administered to students. Those at level 2 cannot perform proofs in the strict sense of the term; those at level 3 can perform informal proofs with verification/communication or explanation functions, while those at level 4 can perform formal proofs that can also incorporate functions such as systematization and discovery (De Villiers, 2004). Based on this idea, and using dynamic geometry software, Lee (2015) showed how pre-service teachers at different Van Hiele levels performed proof tasks that highlighted different interpretations of proof functions. For example, level 4 pre-service teachers understood proof as explanation, discovery and deductive verification functions, while level 3 pre-service teachers only viewed it as explanation and inductive verification.
Bearing in mind that we are working with geometrical proofs, the graphic part plays an important role. When a picture accompanies the answer to a geometrical problem, there can be two cases (Mesquita, 1998); the picture can be seen as an object or as an illustration. If the picture is seen as an object, its attributes or properties can be used in the reasoning of the answer. If the picture is seen as an illustration, it is not always possible to know which theoretical object represents. The author elaborates on this idea showing a picture of a triangle that could be seen as an ideal triangle with no specific measures (object) or as an illustration of a specific triangle. In the particular case of the proof activities, this distinction can be observed in two different possibilities of the use of the pictures: information-related or perception-related use. When the use is informative, the picture shows only the information given in the statement and the picture is considered an object. When the use is perceptual, the picture shows more information than in the statement and is considered an illustration. There are studies that show the bias of the students towards to sustain their reasoning in their perception of the picture more than in the information that it actually gives (Sandoval, 2009). In this respect, Arnal-Bailera & Oller-Marcén (2020), recently brought to light related problems as taking actual measures in a picture to solve a problem or as assuming specific attributes of it when these were not explicitly stated.

The discursive part plays an important role in the proving process, to describe the development of their discourse, we consider Duval’s theoretical construct about the different modes of expansion of the discourse (Duval, 1995). According to Duval, there are two different modes of expansion of the discourse: accumulation mode and substitution mode. The accumulation mode is characterized by the juxtaposition of independent propositions that could be re-ordered without losing its global meaning. The substitution mode is characterized by a logical progression of propositions that follow a non-modifiable order since one proposition is the conclusion of one of the steps of the discourse and, at the same time, the premise in the following step. In this respect, the written discourse has to progress from the accumulation mode to the substitution mode to finish the proof through a deductive process (Saorín et al., 2019).

**Teaching proof**

Throughout this article, we will concentrate on measuring Van Hiele levels for the process of proof, which is understood as an analytical or theoretical proof in the sense described by Gutiérrez (2005). The presence of proof in teaching has often been valued to both show the need to support mathematical knowledge and to understand the concepts involved (Mariotti,
2006). Some studies have delved into the tasks that may be involved in constructing a geometry proof in the classroom or in assessing the construction of the proof (Martin & McCrone, 2003) and how to implement them in class (filling gaps in a proof, conditional statements, local deductions, tests with hints, synthetic proofs with no help, analytical proofs with no help).

Several European countries established the importance of proof in their educative laws. The Organic Law for Improving the Quality of Education (LOMCE, Jefatura de Estado, 2013) states that learning proof during the Spanish baccalaureate (grades 11-12) is compulsory and cuts across all contents. In particular, explicit references to proof include aspects such as the teaching of several methods (reductio ad absurdum, induction, etc.), reasoning (both deductive and inductive) and proof languages (graphic, algebraic or report). In addition, the current Italian educative law, Good School (La Buona Scuola), establishes from 2015 onwards that in the first two years of high school (grades 9-10) students have to “understand the logical steps of proofs and construct simple proofs”. The National curriculum in England for the key stage 4 (grades 10-11) asserts that pupils on this level should “look for proofs or counter-examples; begin to use algebra to support and construct arguments {and proofs}” (Department of Education, 2014).

In a non-European context, the most concrete and complete guide of mathematics teaching is the “Principles and standards for school mathematics” of the National Council of Mathematics Teachers (NCTM, 2000). In particular, concerning proof, the (NCTM, 2000) asserts that “the repertoire of proof techniques that students understand and use should expand during the high school years” (p. 345). More concretely, attending to the proof in geometry the Common Core State Standards (2010, p.76) includes the proof of geometric theorems concerning lines and angles, triangles and parallelograms, this document “allows teachers to be proficient at decision making about what students know, need to know, and how they can impart that knowledge” (Columba & Stotz, 2016). Besides, the (NCTM, 2000) describes what kind of proofs should appear in school mathematics concreting that students “should be able to produce logical arguments and present formal proofs” (p. 345). These sentence clearly reflect that formal proofs (4th Van Hiele level) should be developed in school mathematics.

METHOD AND SAMPLE

The experiment was carried out with 25 students of the Masters’ Degree in Secondary School Teaching at the University of Zaragoza who responded to the questionnaire for two hours. Fifteen of the students have university degrees in mathematics, six in physics, two in
engineering and two in statistics. Concerning their teaching experience, five stated that they had no experience, eighteen had given private secondary classes and two had extensive experience as university tutors.

<table>
<thead>
<tr>
<th>Item</th>
<th>Van Hiele levels</th>
<th>Identification</th>
<th>Definition</th>
<th>Use</th>
<th>Stating</th>
<th>Classification</th>
<th>Proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>●</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.1</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.2, 6.3</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Levels and processes assessed by each item of the questionnaire (Gutiérrez & Jaime, 1995).

The questionnaire we used (Gutiérrez & Jaime, 1995) covers different processes through activities involving polygon properties. It has four open-ended items –items 5, 6, 7 and 8 focusing on the proof–, each with several sub-items (see Table 2). The proof functions (De Villiers, 1993) present in the test are, essentially, verification or conviction functions. The function of the last question is systematization since it underscores the idea of equivalence between definitions. Tasks for assessing geometric reasoning are proofs with hints and unsupported proofs (Martin & McCrone, 2003).

Our study’s design follows a mixed method of the explanatory type (Creswell, 2012) since the qualitative analysis follows the quantitative one. Fundamentally, the techniques used in the quantitative analysis are statistical (cluster analysis) and the answers to the questionnaire are studied in depth in the qualitative analysis.

To decide on the degree of acquisition of each reasoning level in every student, we followed the calculation methodology devised by Gutiérrez et al. (1991) based on the fact that several experts in mathematics education consider the features of mathematical reasoning shown in the questionnaire tasks to be more important than the mathematical correctness of the answers. For
every student’s response to a certain item, a 3-tuple is obtained indicating the degree of acquisition, as a percentage, of the Van Hiele levels that this item measures (levels 2, 3 and 4). The elements in these 3-tuples were agreed between the researchers following Gutiérrez et al. (1991) indications. In particular, we assigned a certain degree of acquisition to every single response (see Table 1), to do so we decided the most accurate descriptor from the list given in Gutiérrez et al. (1991). Finally, we found the mean of the values obtained by each student in each level to calculate his or her degree of acquisition.

The quantitative analysis includes the construction of clusters using SPSS. These clusters are obtained from the three acquisition variables. In addition, other context variables (studies and teacher experience) have been used only to describe them. To construct clusters, we apply the K-means algorithm with the squared Euclidean distance. The number of conglomerates is determined by the Hartigan criterion (Xu et al., 2016).

We conducted a qualitative analysis to characterize each of the previous clusters. In order to achieve that goal, we analyzed the answers of the students attending to the different variables described in the theoretical framework: the use of pictures, the proof scheme or the type of discourse. In addition, other emergent variables inform about specific characteristics of the proof. All these variables have been classified (Table 3) into three different groups informing about general, graphic and proof characteristics.

<table>
<thead>
<tr>
<th>Groups</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>General characteristics</td>
<td>Length of the answer</td>
</tr>
<tr>
<td></td>
<td>Completion rate</td>
</tr>
<tr>
<td>Graphic characteristics</td>
<td>Number of pictures</td>
</tr>
<tr>
<td></td>
<td>Use of pictures</td>
</tr>
<tr>
<td>Proof characteristics</td>
<td>Proof scheme</td>
</tr>
<tr>
<td></td>
<td>Reference to previous results</td>
</tr>
<tr>
<td></td>
<td>Justification of the use of a previous result</td>
</tr>
<tr>
<td></td>
<td>Argumentation grounding</td>
</tr>
</tbody>
</table>
Use of properties that are consequence of the result
Type of discourse (non-substitutive features)
Sensibility to double implication

Table 3. Variables.

There are two general characteristics: length of the answers (number of words) and its completion rate (percentage of the students reaching to a conclusion at the end of every sub-item where unsupported proofs are asked).

There are two graphic characteristics, number and use of the pictures. Concerning the use of pictures, we follow the ideas of Mesquita (1998) and establish several categories: exploratory examples, information and perception.

In the corresponding Tables it is shown the percentage of each use in the items where unassisted proof is asked (5.1, 6.1, 7.A and 8). For the rest of the sub-items (assisted proof tasks) the use of pictures is not considered relevant since the type of hints presented could induce it.

<table>
<thead>
<tr>
<th>Categories (based on Mesquita, 1998)</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exploratory examples (E)</td>
<td>Pictures showing particular cases not explicitly connected to the written argumentation</td>
</tr>
<tr>
<td>Information (I)</td>
<td>Pictures showing only the conditions or attributes established in the statement and explicitly connected to the written argumentation</td>
</tr>
<tr>
<td>Perception (P)</td>
<td>Pictures showing particular cases or attributes non established in the statement that are explicitly connected to the written argumentation</td>
</tr>
<tr>
<td>Without Pictures (WP)</td>
<td>Answers without pictures</td>
</tr>
</tbody>
</table>

Table 4. Categories of the variable “use of pictures”

As we can see in Table 3, we distinguish six different proof characteristics. Based on the work of Harel and Sowder (1998) we have classified the proof schemes into empirical (E), analytic
(A) and responses in which characteristics of both types appear (E/A). There are some non-evaluable answers (NE) due to its little content or because the student left them blank.

Other variables have emerged informing about specific characteristics of the proof: in every sub-item, we studied if there were references to the mathematical results that underpin their argumentations, showing in the tables the corresponding percentages. Other specific characteristics appear only in one item. On item 5, the type of grounding: geometrical (based on arguments of geometric nature), numerical (based on the searching of the regularities in a numeric series) or mixed (characteristics of geometrical and numerical grounding appear). On item 6, we consider the (inappropriate) use of attributes that are consequence of what it is supposed to be proven. This variable refers to the percentage of tasks in which the students use as hypothesis consequences of the thesis to be proved. On item 7, the justification of the appropriateness of the use of previous results. Finally, on item 8, we found two different variables: the appearance of non-substitutive features and the sensibility to double implication. The study of the percentage of students showing non-substitutive features allow us to distinguish different types of discourse. The variable sensibility to double implication shows the awareness of proving the double implication to establish the equivalence of two definitions; this variable expresses the percentage of students making the proof by double implication.

RESULTS

In this part of our work, we present the quantitative analysis, which leads to the construction of the clusters and the qualitative analysis, which leads to their characterization.

Quantitative analysis / Clusters’ construction

After marking the answers to the test, we observed that the Cronbach Alpha coefficient was higher than 0.7 (0.836), therefore we considered that it was a reliable test.

Cluster construction

We will be using 3 clusters in our analysis since, attending to the Hartigan criterion, we have obtained $H(2)=15.62$ and $H(3)=7.10$. The three clusters gather, respectively, 6, 11 and 8
individuals. In Table 5 we show the degree of achievement of each Van Hiele level (Gutiérrez et al., 1991).

<table>
<thead>
<tr>
<th>Cluster 1</th>
<th>Cluster 2</th>
<th>Cluster 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Average score</strong></td>
<td><strong>Degree of achievement</strong></td>
<td><strong>Average score</strong></td>
</tr>
<tr>
<td>Level 2</td>
<td>100</td>
<td>Full</td>
</tr>
<tr>
<td>Level 3</td>
<td>100</td>
<td>Full</td>
</tr>
<tr>
<td>Level 4</td>
<td>74.17</td>
<td>High</td>
</tr>
</tbody>
</table>

Table 5. Average score and degree of achievement of each level for each cluster.

Cluster 1 (C1 from now on) gathers 21% of the individuals (6), they have achieved the four Van Hiele levels, though they only reached a high degree of achievement of level 4. In this cluster, there are 2 graduates in Physics and 4 in Mathematics. Cluster 2 (C2 from now on) includes 44% of the individuals (11), they show full achievement of levels 2 and 3 but only a medium degree of level 4. In this cluster, there are 2 graduates in Physics, 7 in Mathematics and 2 in Engineering. The third cluster (C3 from now on) gathers 32% of the individuals (8), they have a high degree of achievement of level 2, a low degree of level 3 and none of level 4. In this cluster, there are 2 graduates in Physics, 4 in Mathematics and 2 in Statistics.

**Differences between clusters**

Shapiro-Wilk test showed that none of the variables "degree of achievement of the level" follows a normal distribution. In addition, when doing the Shapiro-Wilk test for each level separately, we observed that there was, at least, one cluster not following a normal distribution in each level, thus we rejected the hypothesis proposing that the degree of achievement of each level follows such a distribution. As a result, we apply the Kruskal-Wallis nonparametric analysis leading to the conclusion that there is at least one cluster with significant differences regarding the degree of achievement in each level. In order to identify the statistically different means we applied the Mann-Whitney test for the pairs study concluding that there are statistically significant differences between the level of acquisition of the Van Hiele levels for every cluster and level except for level 2 in clusters 1 and 2 where both present a full level of acquisition. The influence of the variable "teaching experience" was
studied too and it neither affected the composition of the clusters nor was statistically different between the diverse clusters.

**Qualitative Analysis / Clusters’ characterization**

**Item 5**

Item 5 includes two sub-items (5.1 and 5.2). Sub-item 5.1 asks to deduce the formula for the number of diagonals of a polygon given the number of sides and to prove it. This sub-item offers no hints for the task. Sub-item 5.2 asks to deduce the same formula using two particular cases (n=5 and n=6) and its generalization; this sub-item asks for a justification rather than a proof. This item focuses on some of the proof functions: verification (if the formula is proved by induction), conviction (if is proved through a deductive process) and discovering (sub-item 5.2 when it is suggested to find the general formula using particular cases). These functions are to be carried out through an unsupported proof (5.1) and a proof with hints tasks (5.2). This item allows distinguishing the students at levels 2, 3 and 4.

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Variable</th>
<th>Variable</th>
<th>Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>General</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>characteristics</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of the</td>
<td>123,8</td>
<td>97,3</td>
<td>47,7</td>
</tr>
<tr>
<td>answer (item 5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Completion rate</td>
<td>100%</td>
<td>72%</td>
<td>50%</td>
</tr>
<tr>
<td>(5.1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of</td>
<td>3,5</td>
<td>4,2</td>
<td>6</td>
</tr>
<tr>
<td>pictures (item 5)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uses of the</td>
<td>E:33,3%</td>
<td>E:27,3%</td>
<td>E:12,5%</td>
</tr>
<tr>
<td>pictures (5.1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reference to</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>previous results</td>
<td>Not apply</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Argumentation</td>
<td>G:100%</td>
<td>G:40%</td>
<td>G:0%</td>
</tr>
<tr>
<td>(5.1) grounding</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mx:0%</td>
<td>Mx:50%</td>
<td>Mx:37,5%</td>
<td></td>
</tr>
<tr>
<td>N:0%</td>
<td>N:10%</td>
<td>N:62,5%</td>
<td></td>
</tr>
</tbody>
</table>
Table 6. Characteristics observed in item 5.

The answers to item 5 are longer (see Table 6) in the case of students of clusters C1 and C2 (123.8 y 97.3 words respectively) than in C3 (47.7 words). This is related, but not only, with the completion rate that is much higher in students of C1 (100%), while only 50% of the students in C3 finished the tasks. The uncompleted task was the unassisted construction of the formula while every student completed the assisted task (although some of them with errors).

With respect to the pictures used by the students in item 5, we found a significantly higher (at 90%) in C3 students’ answers (6 by student on average) than in C2 (4,2 by student on average) or in C1 (3,5 by student on average). It is remarkable that 50% of the students of C1 and 27,3% of C2 did not make any picture at all when answering 5.1. In C1, the pictures were exploratory examples of the situation to study but not directly supporting the proof itself. In C2, some students (27,3%) made deductions for their argumentation based in the perception of the picture, more than in the information that this picture could actually transmit. 87,5% of the students in C3 acted the same way.
Concerning the specific characteristics of the proof, in this item there was no room to refer to previous mathematical results such as theorems or propositions. We have not founded any examples of accumulative discourse, no matter the different types of grounding they use on its correctness. Attending to the grounding of the arguments used, these are mostly of a geometrical nature in C1 (See Figure 1-left: “Each vertex is ‘connected’ by diagonals with the rest of the vertices but the adjacent ones and itself, thus we have n-3 diagonals at each vertex.”). Arguments combining the geometrical and the numerical ones showed up in C2 (See Figure 1 above-right, instead of multiplying by n the student states at the end of his/her explanation that “(…) If we follow by the other vertices, the number decreases to one.”). In C3, most of the arguments are of a numeric nature (See Figure 1 below-right: the student looks for numerical regularities observed counting the total number of diagonals in particular cases). Some of the
students in C2 that had given mixed arguments in 5.1, gave only geometric arguments in 5.2. In C3, some of the students moved from numerical arguments in 5.1 to geometric arguments in 5.2. In C1, 100% of the students carried out an analytical proof, while in C2, this percentage decreased to 36.4%. In C3, every student showed an empirical proof scheme.

Item 6

Item 6 consists of three sub-items (6.1, 6.2 and 6.3). 6.1 asks to (unsupported) prove that the sum of angles of any acute-angled triangle is 180°. In 6.2 (proof with hints) it is recalled that a pair of parallel lines crossed by a secant form several angles with the same size. Finally, in 6.3 (proof with hints) it is described a complete proof of the statement in 6.1 and it is required to perform similar proofs for straight triangles first and obtuse triangles later. The functions of the proof present in this item are verification/conviction. Sub-item 6.1 enables to classify students at levels 2, 3 and 4 while 6.2 and 6.3 allows distinguishing between levels 2 and 3.

It can be observed that the answers of students in C1 are longer than those given by C2, which at the same time are longer than the ones written by C3. The completion rate observed is very high in C1 and C2 (over 90%), while less than 50% of the students in C3 completed this item.

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Variable</th>
<th>C 1</th>
<th>C 2</th>
<th>C 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>General characteristics</td>
<td>Length of the answer (item 6)</td>
<td>157,2</td>
<td>78,2</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>Completion rate</td>
<td>94,4%</td>
<td>90,9%</td>
<td>45,8%</td>
</tr>
<tr>
<td></td>
<td>Number of pictures (6.1)</td>
<td>3</td>
<td>2,2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Uses of the pictures (6.1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>E:16,7%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>I:50,0%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>P:33,3%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>WP:0%</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Reference to previous results</td>
<td>63,1%</td>
<td>49,4%</td>
<td>25,0%</td>
</tr>
<tr>
<td></td>
<td>Use of properties that are</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>consequence of the result</td>
<td>13,3%</td>
<td>22,7%</td>
<td>66,7%</td>
</tr>
</tbody>
</table>
Table 7. Characteristics observed in item 6.

With respect to the number of pictures used by the students, there are no significant differences in the number of pictures presented by students of the different clusters. However, there exist differences in the use of them since in C1 and C2 its uses are informative or perceptive, 83.3% and 72.8% in the aggregate respectively, while in C3 the main use is as exploratory examples (62.5%).
In 6.1, most of the students from C1 made clear references to the results they were using (see Figure 2 above) whilst references to previous mathematical results are almost non-existent in C2 and C3. In 6.2 and 6.3, it is appreciated that C2 students are more explicit describing the previous mathematical results being used. The explanations given by C3 students have no references to previous results and tend to be limited to a series of computational steps expressed algebraically with very few textual descriptions. Moreover, it has been observed that C1 and C2 students present a better comprehension of the proof techniques since, in general, they do not use properties which are consequence of the result that wants to be proved, but some exceptions can be found (see Figure 2 below-left). This mistake is very common in C3 (66.7%); for instance, in Figure 2 below-right the student uses the sum of the internal angles of a quadrilateral to proof that the sum of the internal angles of a triangle is 180°. In C1 and C2, most proofs follow an analytical scheme, while in C3 in the majority of cases, the answers are blank or with very little content.

**Item 7**

Item 7 includes two sub-items (7A and 7B). 7A asks to prove that two diagonals of any rectangle are congruent while 7B asks to prove that any point in the perpendicular bisector of a segment is equidistant from its endpoints. Both sub-items focus in the verification and discovering functions through unsupported proof tasks. This item allows distinguishing the students at levels 2, 3 and 4.
According to the data shown in Table 8, the answers to item 7 are longer in the case of C1 and C2 students (66 and 58 on average respectively) than in those of C3 (24 words on average). In this item, all the students completed the task meaning they reached a conclusion, even with errors or inaccuracies in some of the intermediate argumentations.

Concerning the graphic characteristics, we can see that every student drew at least one picture and the number of them is higher in C3 than in the other clusters in 7A. However, the use of these pictures is different in every cluster; for instance, some students of C1 used their pictures to face the problem before starting their verbal argumentation, this use has not been identified in other clusters. When changing from C1 to C2 or from C2 to C3 we appreciate that the use of the pictures as information decreases and the use as perception increases. Most of the students of C3 (87,5%) made pictures depicting perceptions against only a 50% of the students.
in C1. Giving more details about these perception-related uses, we can say that in C1, nobody used the same letter to name both diagonals versus 11% and 38% of the students of C2 and C3 who did. In the case of the sides, 60% of the students of C1 named with the same letters versus 86% and 75% of C2 and C3, respectively (see Figure 3 below-right).

We paid attention to the references of previous mathematical results (mainly the Pythagorean Theorem) and if they justified the appropriateness of its use. 100% of the students of C1 using the Pythagorean Theorem in their proof referenced it, this percentage decreased up to 50% and 33.3% in C2 and C3 respectively. Whether they referenced it or not, the fact is that almost every student used it in their item 7 proofs but only 50% of the students in C1 justified the appropriateness of its use (see Figure 3 above, student in C1 claims “Using Pythagorean Theorem $h^2+l^2=r^2$ given $\alpha=90^\circ$”). Only 30% of the students in C2 (see Figure 3 below-left, students in C2 only mentions the Theorem) and none of the C3 students justified the appropriateness of its use. Most of the proofs were classified as analytical regarding its scheme. However, a significant percentage of students in C3 performed an empirical proof.
Figure 3. Examples of students’ answers to item 7 of C1 (above), C2 (below-left) and C3 (below right)

Item 8

Item 8 asks to (unsupported) prove the equivalence between the definitions of parallelogram as “quadrilateral having two pairs of parallel sides” (to what they call usual definition) and “quadrilateral in which the sum of any two consecutive angles is 180°”. In case of a negative answer, it is required to draw a counterexample. The function of the proof shown in this item is systematization. This item allows distinguishing students at levels 3 and 4.

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>Variable</th>
<th>C 1</th>
<th>C 2</th>
<th>C 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>General</td>
<td>Length of the answer</td>
<td>78</td>
<td>39.2</td>
<td>20.5</td>
</tr>
<tr>
<td>characteristics</td>
<td>Completion rate</td>
<td>100%</td>
<td>100%</td>
<td>75%</td>
</tr>
<tr>
<td>Graphic</td>
<td>Number of pictures</td>
<td>2.2</td>
<td>1.7</td>
<td>1.6</td>
</tr>
<tr>
<td>characteristics</td>
<td>Use of the pictures</td>
<td>E:0%</td>
<td>E:9.1%</td>
<td>E:37.5%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>I:50%</td>
<td>I:36.4%</td>
<td>I:0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>P:50%</td>
<td>P:54.5%</td>
<td>P:62.5%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>WP:0%</td>
<td>WP:0%</td>
<td>WP:0%</td>
</tr>
</tbody>
</table>
Responses from students in C1 are clearly longer than in C2, which at the same time are longer than in C3. The completion rate observed is very high in the three clusters being complete on C1 and C2.

Concerning the pictures drawn by the students, it is observed that the number of these in C1 is slightly higher than in the other two clusters. However, the greatest difference appears in their use: in C1 and C2 there is a balance between information and perception uses whilst in C3 the main use is perception and the rest are exploratory examples.

Half of the students of C1 refer to the previous results that are being used while this references decrease in C2 (see Figure 5 below-left, where the student states at the beginning “if we have 2 parallel sides, we can represent it in the following form” without further justification) and are non-existent in C3. Most students in C1 and C2 proof using substitutive discourse; however, in C2 we observe some accumulative features such as unnecessary repetitions of arguments or the proof of the same implication twice. Discourse of C3 cannot be studied due to the scarcity of arguments produced. Concerning the sensibility to double implication, one out of three students in C1 were conscious of the relevance in their use in order to prove the equivalence (see Figure 4-above). Nevertheless, this fact is minority in C2 (18,2%) and does not appear in C3.

Table 9. Characteristics observed in item 8.

<table>
<thead>
<tr>
<th>Specific characteristics</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference to previous results</td>
<td>50%</td>
<td>36,4%</td>
<td>0%</td>
</tr>
<tr>
<td>Non-substitutive features</td>
<td>16,7%</td>
<td>36,4%</td>
<td>NA</td>
</tr>
<tr>
<td>Sensibility to double implication</td>
<td>33,3%</td>
<td>18,2%</td>
<td>0%</td>
</tr>
<tr>
<td>Proof Scheme</td>
<td>E:0%</td>
<td>E:0%</td>
<td>E:50%</td>
</tr>
<tr>
<td></td>
<td>E/A:0%</td>
<td>E/A:18,2%</td>
<td>E/A:50%</td>
</tr>
<tr>
<td></td>
<td>A:100%</td>
<td>A:81,8%</td>
<td>A:0%</td>
</tr>
<tr>
<td></td>
<td>NE:0%</td>
<td>NE:0%</td>
<td>NE:0%</td>
</tr>
</tbody>
</table>

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Figure 4. Examples of students’ answers to item 8 of C1 (above), C2 (below-left) and C3 (below-right).

In C1, and C2, most of the students carry out an analytical proof, whereas in C3 we find a majority of empirical schemes. For instance, the student in Figure 5 below-right bases the proof on the example drawn: he/she only confirms the validity of the statement by copying it and ends the explanation with “Look at the picture”.
Characterization of the clusters

Based on the results obtained for the four items being studied, a characterization of the three clusters of pre-service secondary teachers involved in the study is achieved. In what follows we present a summary of the more relevant characteristics regarding the categories and subcategories analyzed.

Cluster 1

Cluster 1 consists of 6 individuals (24%) of the sample. These individuals show a high level of acquisition of the fourth van Hiele level. They always complete formal proofs without extra help following an analytical scheme. The most frequent use of pictures supporting the responses is informational. Their discourse is always substitutive and their reasoning is mathematically grounded in previous results that are explicitly stated, or at least referred to. These students can operate with the idea of equivalent definitions understanding that a two-way-proof has to be done.

Cluster 2

Cluster 2 consists of 11 individuals (44%) of the sample. These individuals show a medium level of acquisition of the fourth van Hiele level and they have completely acquired all the previous levels. They usually complete formal proofs without extra help following most of the times an analytical scheme. Their discourse is usually substitutive, but showing some accumulative practices. Nevertheless, the most frequent use of pictures is perceptual. Their reasoning is mathematically grounded; however, they do not refer to the previous results in which the reasoning is based.

Cluster 3

Cluster 3 is formed by 8 individuals (32%) of the sample. These individuals show a low level of acquisition of the third second van Hiele level and a high level in the third level. They frequently need hints to start the activities following an empirical scheme in their proofs. The most frequent use of pictures supporting the responses is perceptual. The reasoning shown in their answers is frequently based in numerical patterns or in wrongly deduced properties from the pictures. In most cases, their discourse has not enough content to be described as
accumulative nor substitutive. They do not refer to the previous results in which the reasoning is based and, frequently, they use properties that are consequence of the result that is to be proven.

**DISCUSSION AND CONCLUSION**

With respect to the first objective, “To identify different profiles of pre-service secondary-school mathematics teachers according to the geometric reasoning shown in their proof practices” the use of the questionnaire (Gutiérrez & Jaime, 1995) has been proved adequate. It gave correct consistency values and showed coherence between the answers to each individual item and the levels that such item was supposed to identify. In particular, we considered the items containing proof tasks to analyze their geometrical reasoning. The use of this questionnaire led us to the construction of three statistically different clusters containing 25 PSMTs graduated in Mathematics, Physics, Engineering and Statistics. As Wang and Kinzel (2014) stated, the Van Hiele model gave not enough information to differentiate the specific characteristics of the clusters, what made necessary to delve more deeply into their qualitative characteristics.

With respect to the second objective, “To describe pre-service secondary mathematics teachers proof practices in Geometry”, the analysis of the students’ answers has been based on the study of some variables to explain their main aspects: general characteristics, graphic characteristics and proof characteristics.

In the analysis of the graphic characteristics of the answers, we found an extensive use of pictures with different uses (examples, information or perception). The only exception to this was that some students in C1 did not draw anything to solve item 5 since they could base all the work in mathematical properties. The analysis of the pictures showed differences between clusters: while in C1 the most common use was informative, most of our PSMTs in C2 and C3 used their pictures with perception purposes. Students in C2 and C3 included not only the information written in the given statement but also some other facts that were directly inferred by them without writing down the reasons to do so; these practices prevent geometrical reasoning from developing (Mesquita, 1998; Sandoval, 2009). This could be related with practices observed in pre-service primary school teachers assuming that the illustration accompanying a geometrical problem had an object value to infer conclusions from (Arnal-Bailerà & Oller-Marcén, 2020).
Concerning the proof characteristics of the answers, we worked with Duval’s ideas (accumulation and substitution). All the proofs in C1 and C2 were classified as substitution proofs; however, some of the students showed some non-substitutive features such as the unnecessary repetition of an argumentation. This can be explained since most of our students are Mathematics or Physics graduates and have an important mathematical background which prevented them from having some of the problems shown in similar studies with other PSMTs enrolled in degrees with less mathematical contents (Demiray & Işiksal, 2017; Uğurel et al., 2015). As we delved into the analysis of the answers, we needed to detail the different levels of substitution that our students achieved: most of C1 students made reference to previous results, the percentage decreased as we moved to C2 and C3; however, when the result to mention was a well-known Theorem (Pythagoras) only half of C1 students justified its use and none of the C3 students did so. In this regard, C1 contains the students with a deeper understanding of the proving processes, including ideas as the need of proving the double implication to state the equivalence between definitions, and showing a more formal use of previous results. The students in C2 support their proofs in a more informal use of mathematical properties, showing conceptual mistakes as the use of properties in the process of proof that are actually a consequence of what they were proving. In this respect, Stylianides et al. (2007) showed that most PSMTs struggle with the logic rules involved in the equivalency of two statements and Uğurel et al. (2015) found that, frequently, PSMTs fail to define logical structure of the statements in the theorems. Finally, C3 is formed by students having difficulties to carry out the given proofs and, in many occasions, get to progress on them backing their reasoning in the perceptions of their pictures and in numerical regularities over the mathematical results or the properties of the mathematical objects involved in the proof. The mere existence of this cluster is a source of concern for us, given their background degrees, and make us agree with Karunakaran et al. (2014) that propose a medium- to long-term work to improve PSMTs abilities when these cannot even construct a generic example proof.

Regarding the different types of proof (Harel & Sowder, 1998), our data showed that in C1 most of the proofs were analytical while in C3 most of the proofs were empirical. In C2 we found similarities with both: the proofs of item 5 were empirical while the others were mainly analytical. The extensive use of empirical proofs by students in C3 suggest that they share some characteristics with the undergraduate pre-service middle school mathematics teachers studied by Demiray and Işiksal (2017) or Makovski (2020) that preferred examples over mathematical proofs.
Official documents pay attention to the teaching of proof from a formal point of view, especially in the last years of secondary school. The NCTM (2000) establishes that high school students “should develop a repertoire of increasingly sophisticated methods of reasoning and proof” (p.342). The Spanish’s curriculum official contents (LOMCE, Jefatura de Estado, 2013) correspond with Van Hiele level 4 since this document compels to teach formal proof methods. These curriculums are really challenging for both teachers and students and require teachers with a strong understanding of the mathematical argument (Makovski, 2020). In our opinion, official curricula should explain more precisely the desired level of the different mathematical processes that students should acquire. This concreteness will make it possible to determine the desired level for the PSMTs what, in the end, could give us ideas to improve the education of these future teachers. According to our data, students in C1 showed the appropriate level to develop the official curriculum while students in C2 could find some difficulties and students in C3 have a level lower than the expected in their own students and the required to develop the aforementioned contents in order to make them progress from level 2 to the following levels. These results are consistent with Pandiscio and Knight (2010) who showed that most of the PSMTs were statistically under level 4. It is obvious that, if the teacher cannot proof, it would be difficult for him to teach the different proof methods to his students. Moreover, according to Demiray and Işiksal (2017) and Sears (2019), this weakness in the abilities of the (future) teachers could lead to the avoiding of proofs and the discussions about concepts and relations between them or to the relying solely on the textbook as the expert for how to write the proof.

Acknowledgments The authors wish to thank the referees for their detailed and insightful comments and suggestions that fostered interesting reflections and substantially improved the paper.

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REFERENCES


Hello Problem Solvers, solutions to **Problem 1** were submitted, and I am glad to report that they were correct and interesting. The solutions followed different approaches, which I hope will enrich and enhance the mathematical knowledge of our community.

**Solutions to Problem 1 from a Previous Issue**

**Interesting Square Problem**

Proposed by Ivan Retamoso

**Problem 1**

Let $ABCD$ be a square with side length 1 meter and let $M$ be the midpoint of the segment $AB$. While connecting $D$ and $M$, draw a perpendicular segment from point $C$ to the segment $DM$. 
Marking as \( P \) the point of the intersection of the perpendicular and the segment \( DM \). Find the length of the segment \( PB \).

Solution 1 by Aradhana Kumari, Borough of Manhattan Community College.

This solution is mainly based on Trigonometric properties such as The Law of Cosines, together with Tangent Inverse, it is important to notice that this solution does not require auxiliary lines, which makes it appealing to readers which consider auxiliary lines hard to “figure out”, additionally, the “exact form” of the solution could lead to a general formula for the length of the segment \( PB \) for an arbitrary side length of a given square.

**Solution:** From the right triangle MAD,

\[
\tan \angle ADM = \frac{\text{length of side } AM}{\text{length of side } AD} = \frac{1/2}{1}
\]

Therefore \( \angle ADM = \tan^{-1}(1/2) \)

hence \( \angle PDC = 90^\circ - \tan^{-1}(1/2) \)…(1)

Now consider the right triangle DPC,
\[ \angle PDC = 90^\circ - \tan^{-1}(1/2), \]

\textit{hence} \[ \angle PCD = \tan^{-1}(1/2), \]

\[ \angle PCB = 90^\circ - \tan^{-1}(1/2) \ldots (2) \]

Again, consider the right triangle DPC

We have \[ \sin \angle PDC = \sin (90^\circ - \tan^{-1}(1/2)) = \frac{\text{length of side PC}}{\text{length of side DC}} = \frac{\text{length of side PC}}{1}, \]

Therefore, the length of side PC = \[ \sin (90^\circ - \tan^{-1}(1/2)) \ldots (3) \]

Now consider the triangle PCB, using the cosine rule,

\[ (\text{length of side PB})^2 = (\text{length of side BC})^2 + (\text{length of side PC})^2 - 2 \times (\text{length of side BC}) \times (\text{length of side PC}) \times \cos \angle BCP, \]

therefore, Length of side PB =

\[ \sqrt{(\text{length of side BC})^2 + (\text{length of side PC})^2 - 2 \times (\text{length of side BC}) \times (\text{length of side PC}) \times \cos \angle BCP} \]

\ldots \ldots \ldots (4)

Substituting the value of length of side BC (= 1),

length of side PC = \[ \sin (90^\circ - \tan^{-1}(1/2)) \], (from equation (3)),

and measure of \[ \angle BCP = 90^\circ - \tan^{-1}(1/2) \] (\textit{from equation 2}).
in equation given by (4) we get length of side PB

= length of segment PB

\[ = \sqrt{1 + \left[ \sin\left(90^\circ - \tan^{-1}\left(\frac{1}{2}\right)\right) \right]^2 - 2 \times \sin\left(90^\circ - \tan^{-1}\left(\frac{1}{2}\right)\right) \times \cos\left(90^\circ - \tan^{-1}\left(\frac{1}{2}\right)\right)} \]

\[ = \sqrt{1 + \left[ \sin\left(90^\circ - \tan^{-1}\left(\frac{1}{2}\right)\right) \right]^2 - 2 \sin\left(90^\circ - \tan^{-1}\left(\frac{1}{2}\right)\right) \times \cos\left(90^\circ - \tan^{-1}\left(\frac{1}{2}\right)\right)} \]

\[ = 1 \]

Solution 2 by Aradhana Kumari, Borough of Manhattan Community College.
This second solution uses Geometric properties such as the theorem of Pythagoras and two auxiliary lines together with Heron’s formula for the computation of the area of a triangle, this solution serves to show that different paths can takes us to the same truth.

Solution:

Consider the right triangle DAM, we have AM = \( \frac{1}{2} \) and DA = 1

Since \((AM)^2 + (DA)^2 = (DM)^2\)

Hence \((DM)^2 = \left(\frac{1}{2}\right)^2 + (1)^2 = \frac{1}{4} + 1 = \frac{5}{4}\)

Therefore \(DM = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}\) \(\text{…..(1)}\)

Consider the right triangle CBM, we have

\[ CM = \sqrt{(1)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}\]

Consider the triangle DMC, DM = \(\frac{\sqrt{5}}{2}\), CM = \(\frac{\sqrt{5}}{2}\), DC = 1
Since Area of the triangle DMC

\[ \frac{DM \times PC}{2} = \sqrt{S (S - a) (S - b) (S - c)} \quad \ldots \quad (2) \]

where \( S = \frac{(a+b+c)}{2} \),

\( a = \) length of side DM
\( b = \) length of side CM
\( c = \) length of side DC

\[ S = \frac{\left(\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2} + 1\right)}{2} = \frac{1 + \sqrt{5}}{2} \]

Substituting the value of \( S, a, b, \) and \( c \) in equation given by (2) we get

\[ \frac{DM \times PC}{2} = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{1+\sqrt{5}}{2} - \frac{\sqrt{5}}{2}\right) \left(\frac{1+\sqrt{5}}{2} - \frac{\sqrt{5}}{2}\right) \left(\frac{1+\sqrt{5}}{2} - 1\right)} \]

\[ = \sqrt{\left(\frac{1+\sqrt{5}}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{\sqrt{5} - 1}{2}\right)} \]

\[ = \sqrt{\left(\frac{1}{16}\right)(\sqrt{5} - 1)(\sqrt{5} + 1)} \]
\[
\frac{1}{4} \sqrt{\left(\sqrt{5} - 1\right)\left(\sqrt{5} + 1\right)}
\]
\[
= \frac{1}{4} \sqrt{(\sqrt{5})^2 - (1)^2} = \frac{1}{4} \sqrt{4} = \frac{2}{4} = \frac{1}{2}
\]
\[
\frac{DM \times PC}{2} = \frac{1}{2}
\]
\[
DM \times PC = 1
\]
\[
PC = \frac{1}{DM} = \frac{1}{\sqrt{\frac{5}{2}}} = \frac{2}{\sqrt{5}}
\]

Consider the right triangle DPC, we have \((DP)^2 + (PC)^2 = (DC)^2\)

Therefore \(DP = \sqrt{(1)^2 - \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{1 - \frac{4}{5}} = \frac{1}{\sqrt{5}}\)

Since \(DM = DP + PM\)
\[
PM = DM - DP
\]
\[
= \frac{\sqrt{5}}{2} - \frac{1}{\sqrt{5}} = \frac{5 - 2}{2\sqrt{5}} = \frac{3}{2\sqrt{5}}
\]
\[
PM = \frac{3}{2\sqrt{5}} \quad \text{(3)}
\]
Draw a perpendicular from P on the segment AM, where it intersects the segment AM call it Q.

Consider the right triangle DAM and right triangle PQM, since they both have the same acute angle, angle DAM (or angle PMQ) hence right triangle DAM and right triangle PMQ are similar triangle,

Therefore \( \frac{DA}{PQ} = \frac{DM}{PM} \)

Hence \( PQ = \frac{DA \times PM}{DM} \)

substituting the value \( DA = 1, PM = \frac{3}{2\sqrt{5}} \) (from (3)), \( DM = \frac{\sqrt{5}}{2} \) (from (1)) we have

Hence \( PQ = \frac{1 \times \frac{3}{2\sqrt{5}}}{\frac{\sqrt{5}}{2}} = \frac{3}{2\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{3}{5} \)

\( PQ = \frac{3}{5} \) ….. (4)

Consider again the similar triangle DAM and PQM

We have \( \frac{DA}{PQ} = \frac{AM}{QM} \)
Therefore QM = \( \frac{AM \times PQ}{DA} \), substituting the value of AM = \( \frac{1}{2} \), PQ = \( \frac{3}{5} \) (from (4)), DA = 1, we have

\[
QM = \frac{\frac{1}{2} \times \frac{3}{5}}{1} = \frac{3}{10} \quad \ldots (5)
\]

Since QB = QM + MB, since QM = \( \frac{3}{10} \), and MB = \( \frac{1}{2} \) (since M is the midpoint of the side AB hence MB = \( \frac{1}{2} \))

\[
= \frac{3}{10} + \frac{1}{2} = \frac{8}{10} = \frac{4}{5}
\]

QB = \( \frac{4}{5} \) \ldots \ldots (6)

Consider the right triangle PQB, we have
(PQ)^2 + (QB)^2 = (PB)^2

Therefore PB = \sqrt{(PQ)^2 + (QB)^2} = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1

Therefore, the length of segment PB = 1 meter

Solution 3 by the proposer Ivan Retamoso, Borough of Manhattan Community College.

This solution uses basic facts from Geometry and 2 auxiliary lines, even though it is difficult to figure out the auxiliary lines needed to solve the problem, it shows that, in some cases, basic properties could lead to remarkable results.
Extend $DM$ and $CB$ until they intersect at point $F$. $MB$ is parallel to $DC$, the length of $MB$ is half the length of $DC$, $\triangle MBF$ is similar to $\triangle DCF$ then $B$ is midpoint of $CF$ and $PB$ is the median of the right triangle $CPF$, since in a right triangle the length of the median to the hypotenuse is equal to half the length of the hypotenuse then the length of $PB$ is 1 meter.

Also solved by Jesse Wolf.

I hope you enjoyed solving Problem 1, below is the next problem.

**Problem 2**

The distance between Paul’s home and his School is 2 miles. One day after his school day is over, Paul decides to walk back home by taking 2 straight paths perpendicular to each other, assume the territory where Paul’s home and his school are located allows him to do it, any way he wishes as long as the 2 straight paths are perpendicular to each other.

a) What is the total length of the largest path Paul can take to go back home from his school?

b) Give a compass and straightedge construction of the path you found in part a) starting from the distance between Paul’s home and his School which is 2 miles, using any scale to represent a mile.
Teaching and Learning Processes for Prospective Mathematics Teachers: The Case of Absolute Value Equations

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Abstract: Solving absolute value equations is one of the topics within the course of Selected Topics in School Mathematics (STSM) for prospective mathematics teachers. This research aims to investigate the implementation of the learning and teaching processes of the STSM course for strengthening conceptual understanding and procedural fluency of prospective mathematics teachers for the case of absolute value equations. This qualitative study was carried out through observations on online learning and teaching processes involved 47 mathematics education students, as prospective mathematics teachers, during the Covid-19 pandemic situation. The results of this study revealed that the learning and teaching processes are mainly implemented by using a deductive approach which aided with the use of the GeoGebra software as a tool for helping in the equation solving process and as an environment for developing mathematical concepts. The written student work from the assessment showed various students’ solution methods and difficulties in dealing with absolute value equations. We conclude that the learning and teaching processes of the STSM course need to be improved so as to develop prospective mathematics teachers’ conceptual understanding and procedural fluency in a balanced manner.

INTRODUCTION

Solving absolute value equations is one of the algebra topics within school mathematics that is considered difficult either to learn or to teach (Almog & Ilany, 2012; Çı̇lttaş & Tatar, 2011; Stupel, 2012; Stupel & Ben-Chaim, 2014; Wade, 2012). The difficulty in this topic is not only encountered by secondary school students, but also by mathematics education students as prospective mathematics teachers all over the world (e.g., Serhan & Almeqdadi, 2018; Stupel & Ben-Chaim, 2014), including in Indonesia (Aziz, Supiat, & Soenarto, 2019; Nisa, Lukito, & Masriyah, 2021). Difficulties in solving absolute value equations include, inter alia, determining intervals that make algebraic expressions within absolute value signs positive or negative and applying absolute value
properties during an equation solving process (Aziz et al., 2019; Stupel & Ben-Chaim, 2014). For future careers of mathematics education students, as prospective teachers, these difficulties should be overcome. An endeavor to do so is by strengthening their conceptual understanding and procedural fluency in solving absolute value equations.

One of the courses for prospective mathematics teachers in Indonesia that strengthens conceptual understanding and procedural fluency in school mathematics is so-called the Selected Topics in School Mathematics (STSM) course. Solving absolute value equations is one of the topics within this course. In this course, each topic is addressed by emphasizing conceptual understanding and procedural fluency in a balanced manner. Concerning this STSM course, due to Covid-19 pandemic situation, we wonder how the learning and teaching processes are implemented online so as to strengthen conceptual understanding and procedural fluency of the prospective teachers, particularly for the case of solving absolute value equations.

To investigate online learning and teaching processes of the STSM course, we carried a qualitative study in the form of classroom observations for the case of solving absolute value equations which aided with the use of the GeoGebra software. Previous studies have shown that this type of investigative research, particularly with the use of digital tools in the learning and teaching processes, in Indonesian context, to certain extent, is rarely conducted (Jupri & Sispiyati, 2020; Jupri, Drijvers, & Van den Heuvel-Panhuizen, 2016). Taking this into consideration, the current study aims to investigate the implementation of the learning and teaching processes of the STSM course and its effect toward prospective mathematics teachers’ ability in solving absolute value equations.

THEORETICAL FRAMEWORK

Theoretical frameworks used in this study include types of learning and teaching approaches, didactical functions of technology in mathematics education, and algebraic proficiency. In general, we distinguish types of learning and teaching approaches into two, including inductive and deductive approaches (Prince & Felder, 2006). The deductive approach is implemented by applying deductive thinking in the learning and teaching processes, i.e., teaching mathematical concepts and principles from general to more specific cases (Jupri, Usdiyana, & Sispiyati, 2021; Ndemo, Zindi, & Mtetwa, 2017; Prince & Felder, 2006). As a consequence, the learning and teaching process is carried out consecutively from explaining concepts, definitions, and principles
to using these concepts, definitions and principles in solving exemplified problems; to providing exercises and classroom discussion for students; and to conducting an individual written test.

The inductive approach is implemented by applying inductive thinking in the learning and teaching processes, i.e., teaching mathematical concepts and principles from specific to more general cases (Jupri et al., 2021; Ndemo et al., 2017; Prince & Felder, 2006). Therefore, the learning and teaching process is carried out consecutively from posing specific problems for doing investigations; to constructing conjectures, principles, concepts, or formulas through solving the problems; to applying the concepts, principles, or formulas for solving problems; and to drawing general conclusions.

Regarding the use of technology in mathematics education, Drijvers, Boon and Van Reeuwijk (2010) identified three didactical functions of technology in mathematics education: as a tool for doing mathematics, as an environment for practicing skills, and as an environment for developing concepts. In the function of technology as a tool for doing mathematics, technology users do not need to know and to understand how the technology solves mathematical problems at hand. In other words, the process of obtaining results need not be visible to the users. In this case, technology only serves to help users use time efficiently. For example, when we draw a graph using the GeoGebra software, we need only the result and do not need to know the process of drawing the graph. In the function of technology as an environment for practicing skills, technology plays a role in strengthening users’ skills in performing mathematical procedures. In this function, technology is usually used for solving routine problems. For example, the GeoGebra can be used for solving linear equations in one variable using CAS (Computer Algebra System) feature. In the function of technology as an environment for developing concepts, technology serves to help students in understanding a concept through, for instance, guided investigation process. For example, the GeoGebra can be used as an environment for investigating characteristics of quadratic function graphs.

Concerning algebraic proficiency, it can be interpreted as proficiency in symbolic representation (Brown & Quinn, 2007) which includes conceptual understanding and procedural fluency (Jupri, Sispiyati, & Chin, 2021; Van Stiphout, Drijvers, & Gravemeijer, 2013). Procedural fluency refers to skill in carrying out procedures flexibly, accurately, efficiently, and appropriately; and conceptual understanding means a comprehension of mathematical concepts, operations, and relations (Kilpatrick, 2001). These two aspects of proficiency have to go hand in hand in encouraging proficiency in algebra and in developing algebraic expertise in particular. Algebraic
expertise can be interpreted as an ability that ranges from basic skills such as procedural work with a local focus and algebraic manipulation to strategic work, which requires a global focus and algebraic reasoning and conceptual understanding (Bokhove & Drijvers, 2010; 2012; Drijvers, Godijn, & Kindt, 2010).

METHODS

To investigate the implementation of online learning and teaching processes of the Selected Topics in School Mathematics (STSM) course, we conducted a qualitative study in the form of self-observations. The observations for the case of the topic of solving absolute value equations included two phases. In the first phase, we observed the learning and teaching processes (via Zoom platform) that are implemented in two meetings which lasted for 2 x 100 minutes, and involved 47 mathematics education students in one of the state universities in Bandung, Indonesia. In the second phase, we observed an individual formative written test on solving absolute value equations, which lasted for 30 minutes. After the test, each student should upload his or her answer sheet in an online learning management system of the university, which is called SPOT (Online Integrated Learning System). The first author did the teaching and self-observations on his teaching and learning processes, while the second author checked the observations based on learning and teaching data, including written student work, pictures of the learning and teaching processes, and power point slides with their corresponding additional notes.

Data that we collected in this qualitative study included field notes concerning steps of the learning and teaching processes, lecture notes in the form of power point presentation slides with their corresponding additional notes on the slides, and students’ written work from the formative assessment.

In data analysis, data about online learning and teaching processes were analyzed using the frameworks of types of learning and teaching approaches and of the didactical functions of technology in mathematics education. Data on written student work were analyzed through the framework of algebraic proficiency. In analyzing written student work, we distinguished three methods of solving absolute value equations, i.e., definition, properties, and graph methods. By the definition method we mean the method of solving absolute value equations using the definition of absolute value. By the properties method we mean the method of solving absolute value equations using properties of absolute values. By the graph method we mean the method of solving
absolute value equations using graphs, either produced using a mathematical software, such as GeoGebra, or produced manually by the students.

RESULTS AND DISCUSSION

This section presents the results and discussion of the two phases of observations, including the learning and teaching processes and written student work from the corresponding formative assessment for the case of solving absolute value equations.

Learning and teaching processes for the case of solving absolute value equations

The learning and teaching processes were started by the lecturer through introducing the definition of absolute value for a real number, that is, \(|x| = x\) if \(x \geq 0\) and \(|x| = -x\) if \(x < 0\). In addition, the absolute value of \(x\), or \(|x|\), is interpreted as the distance from \(x\) to 0. For example, \(|3| = 3\) because the distance from 3 to 0 is 3. Similarly, \(|-3| = -(−3) = 3\) because the distance from −3 to 0 is 3. Next, the lecturer used this definition to explain and to obtain some properties of absolute values for \(x, y \in \mathbb{R}\), including \(|x| = \sqrt{x^2}\); \(|x \cdot y| = |x||y|\); and \(|x|/|y| = |x/y|\) where \(y \neq 0\). During the explanation, the lecturer posed relevant questions to students and would continue if the students provided relevant responses.

The lecturer then explained the use of the definition, interpretation, and properties of absolute values for solving absolute value equations. The lecturer gave two examples: (a) \(|x - 1| = 1\) and (b) \(|x + 1| = 2x\). For the first example, the lecturer explained how to solve \(|x - 1| = 1\) using definition, interpretation, and properties of the absolute values. Using the definition, the equation can be written as \(x - 1 = 1\) or \(-(x - 1) = 1\), which lead to \(x = 2\) or \(x = 0\) as the solution of the equation. Using the interpretation, the equation \(|x - 1| = 1\) is interpreted as finding numbers such that their distance to 1 is 1, which leads to \(x = 2\) or \(x = 0\) as the solution of the equation. In addition to this interpretation, the equation \(|x - 1| = 1\) is interpreted as finding abscissas of the intersections of the graphs \(y = |x - 1|\) and \(y = 1\). To do this, the lecturer used the GeoGebra software to show the intersections of the two graphs (see Figure 1). From Figure 1 we could see that the abscissas of the intersections include \(x = 0\) and \(x = 2\), as the solution of the equation.
For the second example, the lecturer explained how to solve $|x + 1| = 2x$ using the definition, interpretation, and properties of absolute values. Using the definition, the equation can be written as $x + 1 = 2x$ or $-(x + 1) = 2x$, which leads to $x = 1$ or $x = -1/3$. By an inspection, $x = 1$ is the only solution for the equation. Using the interpretation, solving the equation $|x + 1| = 2x$ is
interpreted as finding abscissas of the intersections of the graphs $y = |x + 1|$ and $y = 2x$. By using the GeoGebra software, the lecturer drew the two graphs (Figure 2), next he observed that $x = 1$ as the only abscissa of the intersection, and finally he concluded that $x = 1$ as the solution of the equation. Using properties of the absolute values, the equation $|x + 1| = 2x$ can be written as $\sqrt{(x + 1)^2} = 2x$, next both sides of the equation can be squared, be expanded, and be simplified to obtain $3x^2 - 2x - 1 = 0$. The solution for this last equation includes $x = 1$ or $x = -1/3$. By an inspection to the initial equation $|x + 1| = 2x$, it can be concluded that $x = 1$ is the only solution for the equation.

After explaining the two examples above, the lecturer gave an exercise on solving absolute value equations. The tasks for the exercise—screenshot from the power-point presentation—is shown in Figure 3. The first three equations were addressed in the classroom discussion after the students were given sufficient time for solving them. The last four equations were then addressed in the next meeting.

From the discussion of the last four equations, we observed two different solution methods for solving the equation (7), i.e., $x^2 + 2|x| + 3 = 0$, as shown in Figures 4 and 5. Figure 4 shows an example of written student work on solving the equation (7) using the definition method, and Figure 5 presents a written student work using the graph method. It seems that the use of the graph method is a direct consequence of the use of the GeoGebra as a tool for solving mathematics (Drijvers, Boon, & Van Reeuwijk, 2010). After discussing the last four equations, the lecturer then
gave an individual written assessment on solving absolute value equations (addressed in the next subsection).

Figure 4: A written student work using the definition method for solving $x^2 + 2|x| + 3 = 0$

Since $\sqrt{D} < 0$, then the roots of the equation are imaginary. Therefore, there is no $x \in \mathbb{R}$ satisfying the equation.

Figure 5: A written student work using the graph method for solving $x^2 + 2|x| + 3 = 0$

It can be seen from the graph that there is no $x$ satisfying the equation $x^2 + 2|x| + 3 = 0$. 

Dapat dilihat pada grafik bahwa tidak ada nilai $x$ yang memenuhi persamaan $x^2 + 2|x| + 3 = 0$. 

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http://www.hostos.cuny.edu/mtrj/
Based on the description above, we made the following three notes. First, the sequence of the learning and teaching processes for prospective mathematics teachers for the case of solving absolute value equations proceeds consecutively from explaining the definition, the interpretation of the definition, and the properties of absolute values that can be used for solving absolute value equations; explaining the application of the definition and properties of absolute values for solving equations through examples; doing classroom discussions; and conducting an individual written assessment. Considering these processes, which start from general ideas of the definition and properties of the absolute values to more specific ideas of applications, we view that the lecturer used a deductive learning and teaching approach (Bahri, Abrar, & Angriani, 2017; Prince & Felder, 2006). In these processes students were involved actively through a question-and-answer strategy. Therefore, even if the lecturer used the deductive approach, which is recognized as one of the teacher-centered approaches (Ramsden, 1987), the students were still encouraged to participate actively during the learning and teaching processes.

Second, we observed that the GeoGebra is used as a tool for drawing graphs which aids for solving absolute value equations in a more meaningful manner visually. This observation means that the use of technology in the learning and teaching processes includes two functions, namely as a tool for solving problems and as an environment for developing concepts (Drijvers, Boon, & Van Reeuwijk, 2010; Jupri et al., 2016).

Third, the use of the deductive learning and teaching approach aided with the use of the GeoGebra has influenced student thinking in solving equations. In the classroom discussion, we observed some students used the graph method for solving equations, and some other students consistently used definition and properties methods for equation solving processes. This observation is in line with other relevant studies where technology has influenced student mathematical thinking in the process of solving problems (e.g., Bokhove & Drijvers, 2010; 2012; Jupri, Drijvers, & Van den Heuvel-Panhuizen, 2015).

With these three notes of observations, we obtain information about the learning and teaching processes for prospective mathematics teachers and its corresponding qualitative impact on their ability in solving absolute value equations. The ability in solving absolute value equations is further addressed in the next section based on written student work from the formative assessment.

**Analysis of written work on solving absolute value equations**

Table 1 presents findings of written student work on solving absolute value equations from the formative assessment. We view these findings as the effect of the deductive learning and teaching
approach aided with the use of the GeoGebra toward students’ algebraic proficiency. In general, the number of correct solutions for each task is more than about 75%, except for the Task 2 (44.6%). This indicates that the learning and teaching processes worked quite well and seem to have a positive effect to prospective teachers’ ability in solving absolute value equations.

<table>
<thead>
<tr>
<th>Tasks</th>
<th>#Correct solution (%)</th>
<th>Solution methods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>#Definition method (%)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>#Properties method (%)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>#Graph method (%)</td>
</tr>
<tr>
<td>1. $</td>
<td>3x - 2</td>
<td>= 5.$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11 (23.4)</td>
</tr>
<tr>
<td>2. $</td>
<td>x + 2</td>
<td>= 9 - 2x.$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>9 (19.1)</td>
</tr>
<tr>
<td>3. $</td>
<td>1 - 2x</td>
<td>=</td>
</tr>
<tr>
<td></td>
<td></td>
<td>40 (85.1)</td>
</tr>
<tr>
<td>4. $6</td>
<td>x - 3</td>
<td>^2 - 19</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4 (8.5)</td>
</tr>
<tr>
<td>5. $\frac{2x - 3}{3x + 8} = \frac{1}{4}.$</td>
<td>35 (74.5)</td>
<td>37 (78.7)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10 (21.2)</td>
</tr>
</tbody>
</table>

Table 1: Results from data analysis of the written test (N = 47)

Concerning difficulties in solving absolute value equations, we found that the most common difficulties concern checking final results to an initial equation. For example, for the case of solving the equation $|x + 2| = 9 - 2x$, either using definition or properties method, when a student ends up at $x = 11$ or $x = 7/3$, she/he does not check whether each of this value satisfies the initial equation or not. Therefore, the student does not realize that $x = 11$ is not a solution. Figure 6 presents representative examples of written student work for the case of forgetting to check the final calculation to an initial equation of the Task 2. Other difficulties that we found, which are in line with other studies (e.g., Aziz et al., 2019), include difficulties in determining intervals for applying the definition of the absolute value and doing correct algebraic manipulations.

Concerning methods of solving absolute value equations, except for the Task 3, the definition method was used more frequent than the properties method or the graph method. The graph method was only used by some students along with the use of the properties method. This seems that the graph method is used only as a complementary method to ensure the use of the properties method correctly. Figure 7 presents a written student work showing the use of the properties and the graph methods. The more frequent use of the properties method than the definition method for
the Task 3 seems to be caused by the fact that this task is easier to solve by applying properties of absolute values, i.e., by squaring both sides of the equation, expanding each term, and simplifying the whole equation into a quadratic equation. As the use of the definition method mainly depends on the definition of the absolute value, it can be considered that it tends to support the development of procedural fluency of prospective mathematics teachers in solving absolute value equations. As the use of the properties and graph methods needs a comprehensive understanding to equations before executing the solution process, therefore, this can be considered to support the development of conceptual understanding. In line with other relevant studies (e.g., Jupri & Sispiyati, 2020; Jupri, Sispiyati, & Chin, 2021), the findings of this study suggest that procedural fluency is more acquired than conceptual understanding. This might suggest that the algebraic proficiency of the prospective mathematics teachers needs further development to reach the balance between procedural fluency and conceptual understanding.

Figure 6: Representative examples of written student work of forgetting to check the final results to the initial equation
Figure 7: The use of properties and graph methods for solving an equation

CONCLUSIONS

From the description of the results and discussion above, we draw the following three conclusions. First, the observed learning and teaching processes of the Selected Topics for School Mathematics course for the case of solving absolute value equations mainly use the deductive learning and teaching approach aided with the use of GeoGebra and the question-and-answer strategy. In our view, this approach has a strong deductive character because the learning and teaching sequence proceeds from more general ideas, such as explaining definition and properties of absolute values, to more specific ideas of giving examples and explanations on solving absolute value equations. Even if the deductive learning and teaching approach seems work quite well in guiding prospective teachers’ ability in solving equations, still an imbalance acquisition between procedural fluency and conceptual understanding is found. Considering this, we suggest to investigate the use of learning and teaching approaches that provide more opportunities to prospective mathematics teachers to think deeper in understanding of and in solving absolute value equations. This can be carried out, for instance, through providing activities of solving absolute value equations that explicitly request students to use various solution methods more independently. Therefore, the use of well-designed learning and teaching approaches that having inductive and explorative characters seem appropriate to be explored in future research.
Second, the more frequent use of the definition method than the properties method for solving absolute value equations shows that prospective mathematics teachers tend to be supported more on improving procedural fluency than on conceptual understanding. Therefore, for improvement of the learning and teaching processes, we suggest a balanced treatment regarding this, for instance, through putting more emphasize on applying the properties method (when appropriate) in solving absolute value equations.

Third, even if the learning and teaching processes seem work quite well, a number of students still encountered difficulties. These difficulties include making unnecessary mistakes in equation solving process, manipulating algebraic expressions correctly, understanding the meaning of absolute value equations before executing equation solving procedures, and forgetting to check the final calculations to the initial equations. For further investigation, in addition to use appropriate learning and teaching approaches, we suggest to put more emphasize on the use of technology, such as the GeoGebra, as a tool for solving problems and as an environment for developing mathematical concepts. In this way, we expect that the quality of prospective mathematics teachers, particularly in Indonesia, will improve in the future.

In spite of the conclusions above, we acknowledge that this study has some limitations. First, as this study depends on data of observations, field notes, teaching documents, and students’ written work, the data triangulation needs to be enhanced through adding interview data. Through this way, we expect that more comprehensive results on prospective mathematics teachers’ ability and difficulties in dealing with absolute value equations will be obtained. Second, as the observations in this study were carried out from only one cohort of students, we acknowledge that the findings could not be generalized. Therefore, for future research, we suggest to do more extensive observations, including more than one cohort of students, and use appropriate research methods to draw generalization from research findings.

References


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Students’ Difficulty with Problems Involving Absolute Value, How to Tackle this Using Number line and Box Method

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Abstract: Solving problems involving absolute value is one of the hardest topics for students learning in elementary algebra course. This is an important topic in student’s mathematics life since absolute value functions are important example of a function which is continuous on the real line but not differentiable at the origin. A deep understanding of these topics is very important for students. Some students understand the definition of absolute value but are still unable to apply the definition to solve the problems. Others do not understand this concept. This article focuses on using the number line when introducing/explaining this topic. A visual technique called box method helps students to write each step with correct reasoning when solving problems involving absolute value. The number line and box method helps them to see the connection between the concept and procedure used to solve the problems involving absolute value.

Introduction

Solving absolute value equations and inequalities requires students to memorize a series of steps and follow these steps blindly. This article focuses on emphasizing definitions to students rather than just introducing them to symbols in mathematics. The second thing this article focuses that it is very important is that we ask students to critique each step they wrote in their work. This process of asking students to critique helps us as instructors to understand what they know and where we need to focus. In this article, I present some of the students’ work on solving equations and inequalities involving absolute value. For many semesters, I used to teach the absolute value concept as presented in Table 1. I would define $|x|$ as the distance of $x$ from the origin. We read $|x|$ as “absolute value of $x$. I would explain $|x| = 5$ means, we are looking for those numbers whose distance from zero is five units. In the definition and explanation, I did not use the number line which is why I think students struggled to understand this concept. Usually, a book begins a lesson
on how to solve inequalities involving absolute value as shown in Table 1, and I used the same technique as the book does to introduce the definition of absolute value equations and inequalities. The struggle was that students did not understand the absolute value concept. They try to memorize Table 1, and hence they made mistakes. At the end of the semester during the final exam period, none of the students wrote the correct solution for the problem involving absolute value inequality, which made me think to teach this concept differently for better student understanding. I started looking at some articles on absolute value and found many excellent resources (Mark W. Ellis and Janet L. Bryson, 2011). The key idea in these resources was that they all used number line in introducing/explaining the concept of absolute value to students. I also used the number line and a visual technique called “box method” (Kumari, A. 2021) which was helpful for students to understand this concept.

Let c be a positive real number.

<table>
<thead>
<tr>
<th>Equations and inequalities with absolute value symbol</th>
<th>Meaning without using absolute value symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>x</td>
</tr>
<tr>
<td></td>
<td>x</td>
</tr>
</tbody>
</table>

Table 1.

Students’ difficulty

Students do not understand when and why the procedure works (Givvin et al., 2011; Stigler et al. 2010.). Students try to memorize certain “rules” without integrating reasoning and sense-making (Goldrick-Rab, 2007; Hammerman & Goldberg, 2003) and hence sometimes they make mistakes. They used the rote/memorization technique to “understand” absolute value equations and inequalities. They try to memorize the above Table 1 without understanding and hence made
mistakes in solving problems involving absolute values. Using the number line sweeps away the mystery of working with absolute values and empowers students to make connections between procedures and concepts (Mark W. Ellis and Janet L. Bryson, 2011).

Analysis of students’ work

After I introduced the absolute value topic to students, I gave them an assignment. One of the problems of the assignment was solve for $x$ where $|6x - 3| \geq 9$. Below is some of the student work shown for this problem. From the work of students, A, B, C and D, we see that there is a disconnection between conceptual understanding and procedural fluency. Students are unable to translate the inequality involving absolute value to an inequality without absolute value, and hence they are unable to solve the problem correctly. We see that all the students (listed below) made the mistake of not using the keyword “OR” in their procedure. Student A made the mistake at step $6x - 3 \geq -9$ since they did not translate the absolute value inequality correctly and hence rest all the procedure is wrong. Student B made the same mistake as student A when he/she translated the absolute value inequality as $-9 \leq 6x - 3$, another mistake this student made was translating the absolute value inequality “greater than or equal to” as a combined inequality $-9 \leq 6x - 3 \geq 9$. Student C was unable to apply the definition of an absolute value inequality. Student D wrote each step correctly except for the missing keyword “OR”. Students E and F started with a similar procedure, but when I asked them to explain why they wrote $6x - 3 \geq 9$, $6x - 3 \geq 0$; $-(6x - 3) \geq 9$, $6x - 3 < 0$, their explanation was that they found it online and they did not understand the steps.
3. a) Solve for $x$ where

$$|6x-3| \geq 9$$

Solution:

$$\begin{align*}
|6x-3| &\geq 9 \\
6x-3 &\leq -9 \\
6x &\leq -6 \\
6x &\geq 6 \\
x &\leq -1 \\
x &\geq 1
\end{align*}$$

$$\begin{align*}
\text{or} \\
|6x-3| &\geq 9 \\
6x-3 &\geq 9 \\
6x &\geq 12 \\
x &\geq 2
\end{align*}$$

$$x \leq -1 \text{ or } x \geq 2$$
From the analysis of student work

As shown from students works above, we see two kinds of difficulty students face to solve inequalities involving absolute value; firstly, some students do not understand the definition of an absolute value inequality as in the example of the work provided by students A, B, C, E, and F. On the other hand, some students do understand the definition of an absolute value inequality but are unable to correctly write the solution; for example, Student D’s work missed the key word “OR”.

How to help students understand absolute value concept using number line and box

I used number line and a visual technique called box method as explained in “A visual approach to solving equations and inequalities involving absolute value (Kumari. A. 2021). The summary is presented below:
1. Consider Table 1, where the main difficulty for students is that they do not understand how column 1 in the table is related to column 2. Hence, they try to memorize this table and consequently commit errors. We can overcome this difficulty by putting one more column in the middle (as shown in Figure 1) which uses the number line and the definition of absolute value. Figure 1 helps students understand the concept of absolute value and it also helps them to see the relationship between column 1 and column 2 in the Table 1.

2. We replace the algebraic expression inside the absolute value symbol with a box \( \Box \). We transform the problem so that the right-hand side of the inequality is a non-negative real number \( k \), we draw a number line and label \( k \) and \( -k \) on the number line (we draw \( k \) and \( -k \) because these are the two numbers whose distance from the origin is \( k \) units).

\[
\begin{array}{c}
-\infty & -k & 0 & k & \infty \\
\end{array}
\]

\[
\begin{array}{c}
-\infty & -k & k & \infty \\
\end{array}
\]

\( k \) and \( -k \) divide the number line into three pieces (as shown above). Reading from left to right, the first piece consists of numbers less than \(-k\), the second piece correlates to numbers between \(-k\) and \( k \) and the third piece represents numbers greater than \( k \). We find (using the problem) on which piece(s) we keep the box \( \Box \), we write down the piece(s) using the mathematical inequality symbol and the box. Lastly, we replace the box \( \Box \) by the algebraic expression we started with and then solve the inequality.
Absolute Value Equations and Inequalities Interpretation

<table>
<thead>
<tr>
<th>Absolute Value Equations and Inequalities:</th>
<th>Interpretation using the number line and box:</th>
<th>Interpretation without using absolute value symbol:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\Box</td>
<td>=k$</td>
</tr>
<tr>
<td>$</td>
<td>\Box</td>
<td>&lt; k$</td>
</tr>
<tr>
<td>$</td>
<td>\Box</td>
<td>&gt; k$</td>
</tr>
</tbody>
</table>

Figure 1. (Kumari, A. 2021)

Students work using number line and box method

Box method together with the number line helps students understand the concept of absolute value and it also helps them see the relationship between columns 1 and 2 in Table 1. Using the box method enabled students to write the correct procedure to solve problems involving absolute value equations and inequalities. Some student work is shown below. From the explanations of students G and H (shown below), they correctly translated the problem without using the absolute value symbol and wrote each step correctly. Finally, they solved the problem. For student G, the box method was helpful while student H used the number line to tackle this problem.
Conclusion

It is important that we ask students to critique each step they write in their explanation of a problem. This process of asking students to critique helps us understand what they learned and where we need to emphasize. It is very important for instructors to keep looking for the best method to teach as well as help students understand the concepts. Using the number line sweeps away the mystery of working with absolute values and empowers students to make connections between procedures and concepts (Mark W. Ellis and Janet L. Bryson, 2011). The box method helps students to solve problems involving absolute value equations and inequalities with the correct procedure and reasoning (Kumari, A. 2021).

References


How substantial and efficacious is the learning of linear algebra at undergraduate level?

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Abstract: Linear algebra is a crucial part of mathematical training for engineering students and there are many problems in the application of this subject in engineering, physical, and even mathematical disciplines. In this scope, a study on the sustainability of learning this subject and how it can be improved is considered necessary. This article aims to delve into how linear algebra learning is effective and at examining three critical parameters: a) Comparison of summative assessment test results on specific topics with the ones produced in the previous year to assess student learning sustainability. b) Testing the topics based on the approach taken, i.e., based on other mathematical topics and real problems. c) Comparing the effectiveness of learning of problems solved in an analytical fashion and the learning of problems solved with mathematical software's help. The methodology used is based on constructionism that depends on the type of activity given in both the design phase and management phase. From the obtained results of the study, the state of learning linear algebra was measured after one year. From the mathematical retest, it turns out that learning algebra was not effective and the exam grades showed temporary learning. There is a skewed relationship, in the next test those with better grades perform worse than those with lower grades. While in the applied aspect the learning was more sustainable. The scrutiny accentuates the noteworthiness of coordinating the importance of coordinating the semiotic systems for them to produce meaningful and durable learning.

INTRODUCTION

The effectiveness of student learning depends on the individual's willingness and interest in identifying the student's knowledge and the new knowledge that he has acquired. The social
environment, which includes parents, teachers, and stakeholders, can significantly encourage students to pay attention to durable learning. Durable learning alludes to what happens when knowledge and skills remain in the memory bank for a long time (Brown, 2014). Our aspiration as teachers is to remember and apply the knowledge and skills that they have acquired even after the test in their daily practices.

For the most part, student assessment tests serve to indicate the students' appreciation of the course they have attended. The assessment of student learning is considered an integral and fundamental part of the educational activity. In this article, we examine how practical student assessment is in producing good learning. Learning is durable when students can perform in subsequent situations in which the new material is relevant to prior knowledge: student assessment in the subject matter after a year or, is even more significant, in a job in which the mathematics that is supposed to have been learned is to be used. Assessment of learning occurs when teachers use student learning evidence to make judgments about student success against goals and standards. Therefore, the assessment process is connected to two conceptions of learning: learning concerning goals and learning to construct knowledge.

In addition, the article elucidates the question: Is the mnemonic study only useful for exam purposes, or has learning happened? Finally, this article and reclaiming linear algebra learning will also help review the curriculum, the subject matter's initial goals, and the long-term planning.

THEORETICAL FRAMEWORK

Learning is an individual engagement, and as such it is an individual responsibility. The meanings given to reality, on the other hand, can be pooled, compared, and agreed upon (Novak, 2012). Our Guiding Principle is simple: Learning is the consequence of thinking. To comprehend, is to conjecture. To surmise aptly, our students have to be immersed in. Being knuckle down to the knowledge, increases the comprehension and engagement.

Everything we are going to design should help our students stay engaged to pay attention long enough to think effectively and deposit the learning in long-term memory. According to Hermann Ebbinghaus's theory, every time a new concept enters our head, it undergoes a deterioration over time. According to the graph in the figure, we have already lost 40% of what we have learned after twenty minutes. After one day, only 30% of what we have learned is left, and over time the graph flattens out more slowly until only 20/25% is left in our head (Ebbinghaus, 1885).
Consequently, what can incite the more enduring learning to result less scattered but more assiduous?

The retrieval practice, spaced practice, and interleaving elicit the knowledge from long-term memory (Brown et al., 2014; Agarwal & Bain, 2019). Ebbinghaus comprehended that teachers often obliterate that covering only the content does not significantly connote that students have attained it. Covering content does not equate necessarily to mastery of learning, mostly when our students are disengaged and apathetic in our classes (Ebbinghaus, 1885).

Figure 1: Ebbinghaus curve

At this level of inquiry, psychology, neuroscience, and molecular biology closely interact to invoke the students’ learning engagement to be tenacious but not fallacious. The more engaged students are, the more likely they are to recall what they have learned. By definition, student engagement (Ashwin & McVitty, 2015) is that dynamic intersection between being challenged by and loving what one is learning. Think about the times you have been engaged in what you are learning. What
does that engagement encompass? Your head and heart are both engrossed...you are thinking hard and enjoying what you are learning.

Many people believe that their intellectual abilities are ingrained from birth and that failure to meet a learning challenge is an indictment of their birth capacity. However, every time you learn something new, you change your brain - the residue of your experiences is stored. It is true that we begin life with the gift of our genes, but it is also true that we become capable through learning and developing mental models that allow us to reason, solve, and create. In other words, the elements that shape your intellectual abilities lie to a surprising degree within your control. Understanding that this is so allows you to see failure as a sign of effort and a source of useful information - the need to dig deeper or try a different strategy. The need to understand that when learning is hard, you are doing important work.

Neuroscience tells us that the brain believes the hands are the most important part of the body. Making things with our hands stimulates the part of our brain that forms neural pathways and causes synapsis to snap into place. We physically alter our brain by making with our hands. This is why the guiding principle here is that making things visible is one of the best ways for students to learn. The classroom application then is to look for opportunities to have your students "make things" that make visible their understanding of an idea, concept, or term (McQuinn, 2018; Kelleher & Whitman, 2016).

Traditionally, when speaking about evaluation, we refer to particular operations that enhance students' progress. The term "assessment of progress" implies all the operations traditionally carried out by the instructor or an education authority in charge of attending to student activities (Calvani, 2004). Assessment of learning is the process of collecting and interpreting evidence and ultimately summarizing learning at a given time. It helps make judgments about the quality of student learning based on established experience and assigning a value that represents quality (Ontario, 2010).

A search of Brown (2014) has pointed out that there are several unchangeable aspects of learning that all researchers probably need to agree upon. They include:

- Learning to be useful requires memory, which means that what we have learned is still in our memory and can be used later when needed.

Another research (Karpicke et al., 2009) argues that we are poor judges; hence it is hard to judge whether learning is satisfactory or not when we learn well and when we do not. When the going is more challenging and slower and does not seem productive, we are drawn to strategies that seem more fruitful, unaware that the gains from these strategies are often temporary.

Meaningful learning means being able to solve everyday life problems. Problem solving gives purpose to learning, which can only become "meaningful" to the person if they understand its
usefulness and if it serves certain purposes (Jonassen & Rohrer, 1999). Meaningful learning relies on constructivist pedagogy which places special emphasis on the learners, their prior knowledge, and their motivation to learn.

In order to assess how effective a particular course has been or if real meaningful learning has happened, you need to see how well students perform when they are given challenging tasks for which they have been taught in the course, or, at least they are aware of the importance of the task, and its difficulty and but that is no easy task.

Research studies have confirmed that student evaluations are very helpful in predicting the quality of learning (Scott & West 2008). However, other studies have proved the opposite. The correlation between student evaluations and the quality of learning is negative. The higher the instructor's score on student evaluations, the worse the learning. Conversely, when the assessment score is lower, the better the learning (Keith, 2019).

Instructors may increase grades or reduce instructional clarifications to elevate student evaluations. In this case, it is hard to realize how each of these measures correlates with the desired outcome of actual student learning. Standardized tests are not administered at the postsecondary level, and the grades generally cannot assess student academic achievement due to the heterogeneity of assignments/exams and the mapping of these assessment tools applied by individual professors' final grades (Scott et al., 2010). Research has revealed that student evaluations positively predict student outcomes in contemporary courses but are low indicators of student outcomes in subsequent courses (Keith, 2019).

It could be argued that learning occurs when mistakes get corrected by students, so an especially effective way to teach something in a way that sticks. There is no doubt that students will make mistakes, but when at a later stage, they correct the mistakes on their own while taking into account the instructor's suggestion, means that the learning goal has been achieved. Nevertheless, getting everything right is a counterproductive goal in education. Quality learning occurs when error correction happens after some time the error was made (Bjork, 1994).

The practice of retrieval (Roediger et al., 2011; Smith et al., 2016) by recalling facts, concepts, or events from memory - is a more effective learning strategy than revising through rereading. Retrieval strengthens memory and interrupts forgetfulness. A single simple quiz after reading a text or listening to a lecture produces better learning, and better recall, than rereading the text or reviewing lecture notes. Periodic practice arrests forgetting, reinforces pathways to recovery, and is essential for students to focus on the knowledge they want to acquire.
Agarwal (2019) explains "Rather than asking students to retrieve similar types of information in one continuous session, apply the principle of interleaving, which mixes up content from different areas. Students retain and learn more information when they mix it up" (p.22). When information is processed in different ways, establishing more connections across the neural branding network and encoding learning more deeply (Kelleher & Whitman, 2016). In addition to doing regular retrieval practice on material that was covered recently, it's also important to revisit old information, asking students to retrieve information a few days, weeks, or even months after they learned it. Because that information is harder to recall, it actually makes the learning that comes from it that much more durable (Agarwal & Bain 2019).

The quality of learning assessment depends on "what you observe" so that you can verify it: it must refer to a meaningful, non-scholastic assimilation of knowledge. When assessing learning assessment tools that can predict the outcomes of the assessment must be in place, in other words what is the student able to do with the knowledge he has acquired real life situation (Sergiovanni & Starratt, 2003). A research warns that when students carry out a task and spend some time between sessions, or the task is more complex and involves two tasks or subjects simultaneously, error correction is more challenging, while learning is sustainable and the implementation of forthcoming tasks is less challenging (Sternberg & Grigorenko, 2002).

According to Nickerson (1979) when the student is able to establish the principles or the "rules" which differentiate the types of problems, he is more successful in choosing the right solutions in unfamiliar situations. This skill is better acquired when interrelated and varied practices are applied than through the application of group practice. Any new learning requires a foundation of prior knowledge (Callender & McDaniel, 2009). In order for the student to learn numerical analysis, one must remember algebra and mathematical analysis. If the student is only engaged in mechanical repetition, the ability to store all the information in his brain is compromised.

However, elaboration of information is the process of giving new meaning to the new material in which the student uses your own words and connects prior knowledge with new knowledge. The more one can explain how new learning relates to previous knowledge, the stronger the understanding of the new learning will be, and the easier it is to remember and use it in the future (Brown et al., 2014; Callender & McDaniel, 2009; McCabe, 2010).

Assessment of learning is often based on the themes or projects and the judgments are made on the basis of student performance on tasks, and evaluation involves various areas (Comoglio, 2005). So, this process involves:
Information-gathering that occurs at the end of a teaching/learning process for the purpose of verifying whether the learning goals have been met.

METHODOLOGY

In this section, we present the methodology used in designing the activity learning path. The methodology used is based on constructionism that depends on the type of activity given in both the design phase and the management phase. The type of activities in the scheme of the created methodology is based on the given theoretical framework. A descriptive statistic was used to present the data. The transition from one phase of the scheme to another is based on qualitative analysis of data in both phases, management, and design.

The topics that are being tested are part of linear algebra and were taught to students a year ago. The students are sophomores, majoring in engineering in total 30. The students who participated in the study are from the same tutorial group and with the same teacher. 82% of them are male, 18% female, and 3% of participants are of Egyptian ethnicity. The class chosen for the study contained 34 students. The level of the students is average.

Students have been notified about the test and the study a week in advance. As it was a non-examination test the students did well, especially with the application problems and in the use of Matlab.

The activities to be implemented were accomplished according to the following outline presented on Figure 2:
In the first [Examples 1 and 2 in appendices] and third phase [Examples 5, 6 and 7 in appendices], the examination and test 2 include open-ended questions. As a result, at this stage, a summative assessment is conducted. In the second phase, test 1 [Examples 3 and 4 in appendices], includes activities in the form of a quiz with feedback for each answer. Here we have a diagnostic assessment. In the first and second phase, these are cognitive activities. In the third phase, test 2, we render activities that are not only cognitive but for the most part metacognitive. They fall into three parts: real problems from other disciplines, mathematical topics which require students to apply previously learned topics, and in the end, topics deciphered in Matlab software.

AN APPLICATION TO A CASE STUDY

The methodology described in the previous section has been applied to the case of the module of Linear Algebra for engineering undergraduates. Question:

*What evidence did students learning demonstrate?*

Students were asked to produce but not to reproduce knowledge, the case in test 1 and test 2. The tests are connected to the real world and to parts of the mathematics curriculum. Tests take account of student differences; tests that require of the students to apply the analytical skills, and tests in which the mathematical software can be applied. Tests allow for assessment of complex skills and allows students to demonstrate student success in a variety of ways.

*What tools can be used to gather information?*

Summative assessment information should be sought based on tasks/performances that demonstrate the mastery of objectives, essential skills, and competencies required in teaching. To see how effective the learning is, the first parameter in question should be discerned, that is, the comparison of the summative evaluation, the exam, [Example 1 and 2 in appendices] with topics tested after one year [Example 3 and 4 in appendices]. The tests are in a closed version, i.e. different from the tests in the first phase (examination), they are quizzes.

In the exam that was carried in the first phase, the summative assessment of linear algebra is given by the following figure:
Student results are for the most part below average 7, 23. In the event of getting grade 4 the student does not pass the exam. While 10 is the highest grade. From the results of the first test, we identify two issues:

- The report between the exam grade versus the grade in test 1, in the second stage.
- The qualitative results of students in the version of the quiz-type questions, test 1.

The comparison of the exam grade with the test grade 1 in the second phase is given in the figure below:

Grade 4 in the exam becomes 5.74 in test 1, grade 5 becomes 5.33 continuing in the same way. The most stable were the students with grade 7. Then, as it increases, the grade of test 1 (of later) decreases, while with the decrease of grade 7 the grade of test 1 rises. Finally, after one year the
number of students with low grades than with high grades. This is indicative of the quality of learning.

Straightaway, we should analyze the students’ results in the second phase (test 1) by comparing them with the results of the first phase, in other words with the test results of the previous year. The most common errors in the second phase, test 1, are:

- Lack of concepts, algorithms, theorems that are increased compared to the exam test.
- Many students with lower grades do not give feedback to the answers; this might indicate that they are not used to questions in which the answers are closed.
- There is no increase in linguistic errors which might be attributed to the types of the question.
- Very often we observe Harlow’s error factor (they do well but not what they are asked).
- There are contradictions between the feedback and the answers given.
- Reduced number of errors when reading the questions and answers.
- They demonstrate less than in the exam test.
- There are more mistakes than errors (Pepkolaj, 2015).

In the third phase, test 2, the topics that are addressed are the same with the ones in the second phase (test 1) but require students to apply the knowledge in a variety of situations. The topics address problems that relate to:

- Everyday life.
- Disciplines such as engineering, physics, chemistry.
- Other mathematical topics in which the topics of the second phase serve us.
- Problems that can be solved with the help of the Matlab software.

The table below indicated the average grades of the students grouped into the exam tests, test 1 and test 2:
Group 1 includes students who have received grade 4 in the exam test. In test 1 the grade becomes 6 and in test 2 the grade is 6. Group 2 includes students who have received grade 5 in the exam test. In test 1 the grade becomes 6 and in test 2 it becomes 8. If we are analyze the data of the third phase, consequently test 2, in quantitative terms, we observe that students with initial grade 7 (group 4) are more stable. On the other hand, the results of students with grade below 7 in the previous two phases increase but the results of students with grade above 7 do not arrive. It should be noted that the students with the best grade in the exam test, also do better in the third phase test.

However, students with the worst grades in the exam test get better results in test 2 but do not add students with better grades. If we compare test 1 with test 2 the results get better and the results of students with better grades are also maintained. Based on this analysis, we could conclude that more learning occurs in the application stage because previous activities focus more on theory.

We should do a qualitative analysis of the students' results from the third phase. The problems that were given to students to solve focused mainly on the application version. Students were asked to solve them by applying their analytical skills and employing graphical and numerical ways. The aim of this approach was to observe whether the student can easily pass from one solution way to another, from one semiotic representation to another. It ran out in all types of problems the analytical method produces more errors, usually the same as the other phases, but the numerical method and particularly the Matlab software helps students to increase students' performance.

They use Matlab commands and scripts well, but do not write comments. They only solve the problem in the numerical version which was also asked for the analytical one. The most important thing is that the student switches easily from one semiotic representation into another thus making learning sustainable.

One of the crucial objectives of the subject of algebra is the application of this knowledge in solving engineering and practical problems. From the above analysis it was seen that the objectives of the course are partially achieved. Even the program with its objectives needs changes. In the subject of algebra there is no Matlab program but it is done separately and not in the same semester. The connection between assessment and learning is a delicate issue, and when we discuss about effective learning the problem is even more obvious. Generally, we are poor evaluators to when we learn well and when we don't.

CONCLUSIONS AND RECOMMENDATIONS

The parameters brought into play in this article sought to demonstrate when learning linear algebra is stable and meaningful. Comparing the summative assessment by testing topics but in a different way after one year indicates the correlation between them is negative. After one year, more
learning happens, but with the lower grades when contrasted to the higher grades in the previous test. The quality of learning decreases when previous grades measure a momentary learning. However, the approaches/lessons, which they have learned, should remain and be implemented throughout life.

Furthermore, the topics that have been tested in the application version i.e. in the other mathematical topics and real life problems, show that students do better, and that there is no negative correlation between the earlier grades with the later grades. There is more learning in the application version and the best results are achieved when using the mathematical software, which allows the student to move easily from one semiotic representation to another.

However, the transition from the open-ended test of the exam to the close-ended test in the second phase enabled us to see the mnemonic study applied only in exam situations (for exam purposes). Therefore, there is a need to create balanced, well-designed assessment items that accurately measure what students know and are available to do. In conclusion, the learning of linear algebra was not sustainable and was not carried out properly and we suggest a review of the program and objectives of the algebra course. The following suggestions can be given:

▪ A text that includes theories, exercises, engineering problems, Matlab codes of their solution because it is very important to be all together. The student must move freely from one presentation to another through which guarantees us a stable education.

▪ This study suggests that course programs or amelioration should be done based on the difficulties of students (recovery of previous knowledge) and not just based on the labor market. We see this point as essential.

APPENDICES

In this addenda, we encompassed a series of sample questions, illustrating the three phases of testing. Some questions from the first phase (exam) are:

Example 1 Consider the following linear system

\[
\begin{align*}
  x + y + hz &= h + 1 \\
  x + hy + z &= 2 \\
  tx + y + z &= 2h
\end{align*}
\]

Discuss compatibility and calculate possible solutions for \( \forall h \in \mathbb{R} \)

Example 2 Let \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) be the endomorphism such that:

\[
f(x, y, z) = (x + hz, x + hy + 2z, x + y + hz)\]

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\[
f(x, y, z) = (x + hz, x + hy + 2z, x + y + hz)\]
a) Say for which values of h, f is not injective and calculate the dimension and a base B of ker f
b) Say for which values of h, f is not surjective and calculate the dimension and a base B' of Imf
c) Calculate $f^{-1}(3, -1, 1)$, varying $h \in R$

The examination track in the first phase contained six questions in the open version like the examples above, for a total of maximum 100 marks, and there were four tracks. The grade were according to the evaluation of the Bologna system.

Some sample questions from the second phase (test 1) are:

**Example 3** In trying to find out if the lines in 3D space

$$x = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + l \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

and

$$x = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + m \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

have a common point of intersection, we solve a system of 3 equations in the two unknowns l and m. After applying a number of elementary row operation there is yielded

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$  What conclusion can be drawn about the point of intersection?

- The point of intersection is (-1, -3, 0)
- The point of intersection is (1, 3, -1)
- The point of intersection is (-1, -3, 1)
- The two lines do not intersect
- There is not enough information given to answer the question

**Example 4** Let $A \in M_{88}(R)$ be the matrix in which the elements $a_{ij}$ with $i \geq j$ are equal to 88 and the others are null, then $A$:

- has exactly 88 eigenvalues where 44 are real numbers and 44 are complex numbers with non-zero imaginary part
- $\mu=88$ is an eigenvalue of $A$ with multiplicity 88
- has exactly one complex eigenvalue with zero imaginary part
- it has no real eigenvalues
- it has exactly two distinct real eigenvalues

Test track 1 contained 18 questions in the closed version quizzes like the examples above for a total of 100 points and there were three tracks.
Some sample questions from the third phase (test 2) are:

**Example 5** Let us find coefficients of a cubic polynomial \( p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \) that satisfies \( p(1.1) = 2.3, p(-0.3) = 3.3, p(0.4) = -4.2, p(0.7) = 6.1 \)

**Example 6** For example Alka Seltzer makes fizzy soothing bubbles through a chemical reaction of the following type: \( NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2 \)

The reaction above is unbalanced because it lacks weight to describe the relative numbers of the various molecules involved in a particular reaction. In a chemical reaction the atoms that enter the reactions must also yield reactions. How do you write and balance chemical equations?

**Example 7** Consider the electrical network shown in figure 6

![Figure 6: The electrical network](image)

Consequently, the \( R_k \)'s are positive numbers denoting resistances (unit: ohm (Ω)), the \( i_k \) are currents (unit: ampere (A)), and \( V_1 \) and \( V_2 \) are the voltages (unit: volt (V)) of two batteries represented by the circles. Using Kirchhoff's law, assume that \( R_1 = 6 \, \Omega, R_2 = 16 \, \Omega, R_3 = 2 \, \Omega, R_4 = 5 \, \Omega, R_5 = 3 \, \Omega, R_6 = 6 \, \Omega, V_1 = 70 \, V \) and \( V_2 = 61 \, V \). Find the six unknown currents. Test track 2 containing 4 questions in the open version like the examples above for a total of 100 marks, and there were eight tracks.

**References**


Learning trajectory based on fractional sub-constructs: Using fractions as quotients to introduce fractions

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Abstract: Learning emphasizing fractions as a part-whole concept causes several limitations in developing fraction knowledge and inhibits proportional reasoning. We use fractions as quotients as the first context introduced in our learning trajectory. We report the teaching experiment results using the improved learning trajectory on thirty 4th grade students in Jakarta's public schools. The findings of this study indicate that the fractions as quotients used as the first stage in the learning trajectory can lay a solid foundation for the concept of partitioning in a variety of strategies and the concept of fractional parts. Besides, the developed learning trajectory has provided opportunities for students to learn about fractional mental operations, which are interrelated and serve as the basis for the development of proportional reasoning.

Keywords: Fractions subconstructs, partitioning, iterating, unitizing, multiplicative reasoning, proportional reasoning

INTRODUCTION

Fractions are one of the concepts used to solve problems and communicate in everyday life, as not all daily life problems can be translated or solved with integers. For example, when purchasing sugar, sometimes it requires less than a kilo, or when slicing bread, it needs its parts to be shared with some people. Besides, the concept of fractions is essential for students to understand advanced mathematical concepts, such as arithmetic, algebra, geometry and measurement, probability, and statistics (Purnomo et al., 2019, 2017). However, many existing studies in the broader literature...
have indicated that fractions are one of the problematic mathematical content for students, especially at the elementary school level (Charalambous & Pitta-Pantazi, 2007; Purnomo et al., 2019, 2017). The possible reason is that instrumental teaching and learning are still implemented in mathematics classrooms (Purnomo et al., 2014).

As the importance of fractions in daily life is generally known, the curriculum structure and contents should be organized well to help students understand and apply fractions. As a matter of fact, in Indonesian curricula, research-based learning trajectories to construct the concept of fractions have not yet been considered in the teaching and learning process and textbooks (Rahmawati et al., 2020). Clements and colleagues (Clements et al., 2019, 2020) argue that research-based learning trajectories are essential for developing instructional planning, curriculum (textbooks), teaching, and assessment. Therefore, our research focuses on developing a learning trajectory based on fractions subconstructs to promote fractions concept development.

**Fractions Subconstructs: Meanings of Fractions**

Kieren (1976) introduced interrelated meanings or subconstructs of fractions, namely fractions as ratios, fractions as measures, fractions as division (also known as fractions as quotients), and fractions as operators. Behr and colleagues (Behr et al., 1983) added another subconstruct, fractions as part-whole (Lamon, 2007). They established theoretical relationships among the five subconstructs, the basic operations of fractions, fraction equivalence, and problem-solving (Charalambous & Pitta-Pantazi, 2007). The summary of the five subconstructs is presented in Table 1 (Clarke et al., 2011; Lamon, 2007, 2012; Purnomo, 2015; Watanabe et al., 2017; Wilkins & Norton, 2018).

<table>
<thead>
<tr>
<th>Subconstructs</th>
<th>Descriptions</th>
<th>Mathematical Statement</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fractions as part-whole</td>
<td>A situation in which a continuous quantity or a set of discrete objects are partitioned into equal parts</td>
<td>( \frac{m}{n} ) whereas ( m ) is taken from ( n ) with fair sharing.</td>
<td>( \frac{2}{3} ) means two parts out of three equal parts of the pizza.</td>
</tr>
</tbody>
</table>
Fractions as quotient

Any fraction can be seen as the result of a division situation. $\frac{m}{n}$ as dividing a quantity $m$ into $n$ with fair sharing. $\frac{2}{3}$ means two pizzas divided by three people or each person's amount when three people share a 2-unit of pizza.

Fractions as measures

Fractions can represent a measure of the quantity relative to one unit of that quantity $\frac{m}{n}$ as being $m$ measures of the unit fraction, namely $\frac{1}{n}$, or $m$ is iterations of $\frac{1}{n}$. Two pieces of $\frac{1}{3}$ pizzas are vegetables.

Fractions as operator

Fraction as functions apply to some number, object, or set $\frac{m}{n}$, whereas $m$ is applied to the number of $\frac{1}{n}$. $\frac{2}{3}$ of 24 marbles or $\frac{2}{3}$ of glass or $\frac{2}{3}$ kg of sugar.

Fractions as ratio

Fractions as ratios refer to the comparison between two quantities; therefore, it is considered comparative index, rather than a number $\frac{m}{n}$, whereas it is stated the relationship between $m$ and $n$, whereas $m + n = \text{whole (part-part relationship)}$ or $n = \text{whole (part-whole relationship)}$. The case of two green marbles and six red marbles can be interpreted differently. The first case is the relationship of parts to other parts, 2:6 or 1:3 (as the number of green marbles is 1/3 of the number of red marbles or the number of red marbles is three times the number of marbles green). The second case is the relationship of parts to the whole, 2:8 or 1:4 (using the same analogy).

Table 1. The Five Subconstructs of Fractions
As shown in Table 1, the five subconstructs of fractions are interrelated. Each plays an essential role in developing the concept of fractions and other subjects that require the concept of fractions. Referring to the study conducted by Charalambous and Pitta-Pantazi (2007), the part-whole construct is a fundamental concept for developing other subconstructs. An illustration of the relationship is illustrated in Figure 1.

![Figure 1. The five subconstructs of fractions and their relationships](image)

Based on Figure 1, the part-whole subconstruct is at the core of other subconstructs' development. It is also empirically corroborated by the results of Charalambous and Pitta-Pantazi’s (2007) study on elementary school students in Cyprus. However, the relationship between subconstructs remains unclear as there is no further explanation concerning directions that connect the rest subconstruct. It is also in line with Lamon (2007) and Tsai and Li (2017), who argued that the five subconstructs are interrelated. However, a further explanation of the empirical relationship has not been found in the literature. Although many authors have conducted studies, this problem is still insufficiently explored.

**Fractional Mental Operations**

Fractional mental operations refer to fractional mental actions that have been abstracted from experience to become available for use in various situations (McCloskey & Norton, 2009; Norton & McCloskey, 2008; Steffe, 2004; Steffe & Olive, 2010). In this study, we focus on partitioning, iterating, and unitizing.

Previous studies have shown that partitioning and iterating activities determine the foundation for developing fractions knowledge, fractional number sense, and fraction operations (Purnomo,
Partitioning is defined as dividing an object or several objects into disjoint and exhaustive parts (Lamon, 2012). Some used the term of equipartitioning to indicate that the part in question is the same size (c.f. Steffe & Olive, 2010). Meanwhile, iterating refers to a mental process that repeatedly copies specific fraction units to get the whole or other fraction units. Copying parts is repeatedly carried out to get a part of another whole, the whole itself, and more than the whole (Purnomo, 2015; Singh, 2000; Wilkins & Norton, 2018).

Partitioning activity is often identified with the introduction of fractions as part-whole relationships, i.e., determining the part that is the focus of observation of the whole. The whole unit can be discrete (e.g., six candies) or continuous (e.g., a tube). However, this activity can be developed through other subconstructs. An example of fractions as quotients is that when three children are sharing six cookies. It can be represented mathematically as $6 \div 3$ or $6/3$, in which the process of sharing six cookies to three children can involve various possible partitioning strategies, such as distributive partitioning, halving, or recursive partitioning (Lamon, 1996; McCloskey & Norton, 2009; Shin & Lee, 2018; Steffe & Olive, 2010). This activity became an alternative to introduce fractions, where the part-whole subconstruct, which is put first in most curricula, was less successful in laying the foundations of the concept of fractions (Simon, Placa, Avitzur, & Kara, 2018; Watanabe, 2006; Wilkins & Norton, 2018).

Fractions as measures also accommodate partitioning and iterating activities. These partitions and iterations consist of mental activity that can be composed of one another and, more specifically, form inverses of each other to cancel each other out when compiled (Wilkins & Norton, 2018). However, some researchers note that the concept of fractions as measures is more challenging than other subconstructs, especially when it involves using number lines (Charalambous & Pitta-Pantazi, 2007; Izsák et al., 2008).

Partitioning skills can also be developed by unitizing (Lamon, 1996, 2012). Unitizing is a cognitive process for conceptualizing a given quantity as a unit or whole (McCloskey & Norton, 2009; Norton & McCloskey, 2008). For example, setting two isosceles triangles into a single unit using a patterned block. These activities can run simultaneously in developing the concept of fractions, especially part-whole relationships (Pantziara & Philippou, 2012), equivalent fractions, and in turn, developing the concept of fractions as quotients (Lamon, 2012). There are several activities in developing unitizing, one of which asks students to generate equivalent expressions for the same quantity. For example, there are 24 beads, and it will be equivalent to two groups of 12 beads or...
three groups of 8 beads. It can also be developed with fractions as operators because students will be asked to determine the unit before operating it.

Partitioning, iterating, and unitizing activities play an essential role in developing students' multiplicative reasoning (Singh, 2000) and proportional reasoning (Lamon, 1996; Langrall & Swafford, 2000; Purnomo, 2015). Besides, when students learn fractions as ratios, proportional reasoning might be increasingly apparent. The ratio is the core of proportional reasoning and the problem of proportionality (Purnomo, 2015). Langrall and Swafford (2000) revealed that there are at least four essential prerequisites in proportional reasoning, including: (1) developing increasingly complex unitizing, (2) recognizing situations and reasonable or appropriate ratio, (3) understanding two different forms of the ratio which might not necessarily have different values, and (4) recognizing the difference between additive (relative) and multiplicative (absolute) relationships. Furthermore, Purnomo (2015) states that the development of proportional reasoning can be carried out initially by constructing a foundation for understanding fractions, decimals, percentages, ratios and connecting them and related contexts.

**Present Study**

We see that the relationship among subconstructs is complex and dynamic, so which is the best possible subconstruct to introduce fractions? Other researchers also asked this question (Lamon, 2007; Watanabe et al., 2017), and the answer to it requires an empirical study. Other possible questions are: Is it better for students to be exposed to all five subconstructs early, or is it better to focus on one (beyond part-whole)? If it is better to focus on one, which? Do students need to understand all five subconstructs before Algebra? Traditionally, fractions as part-whole were the first to be introduced in most countries’ curricula and textbooks (Rahmawati et al., 2020). However, some researchers criticize it (Simon et al., 2018; Watanabe, 2006; Wilkins & Norton, 2018), and in fact, the curriculum is more focused and only limited to part-whole subconstruct (Rahmawati et al., 2020). Simon et al. (2018) revealed that fractions as measures are more effective than part-whole and provides several benefits. However, studies also explain that students encounter obstacles using number lines (Charalambous & Pitta-Pantazi, 2007; Izsák et al., 2008).

Besides, we agree with the constructivist view that students will construct knowledge easily when exposed to related prior knowledge and experience. Therefore, we started with fractions as a quotient. We use the child's experience by considering the whole number's division to develop the partition concept. Furthermore, Hackenberg (2010) also states that fractions as quotients have not
received more attention in fractions research. In other words, the research literature has not been found that considers division as the starting point for learning fractions.

Similar to previous studies (Sari et al., 2020; Simon et al., 2018; Steffe, 2004), in this article, we solely focus on the teaching experiment findings associated with the revised HLT. In the first HLT, we use fractions as quotients as a starting point, then it is followed by fractions as part-whole, fractions as ratios, fractions as measures, and it ends with fractions as operators. However, in the teaching experiment session, we revised this HLT by placing the fraction as ratios at the end by referring to the findings of the preliminary teaching experiment that this section involves the complexity of other subconstructs, in addition to the cognitive characteristics of students who are not ready for this material and the goal of establishing proportional reasoning. The revised HLT can be visualized in Figure 2.

![Figure 2. The Fractions Subconstructs Learning trajectory](image_url)

Figure 2 illustrates the path through which the research objectives are reached. Each learning session has its sub-goals and supports the primary goal. The objectives of each learning session
are: (1) students can develop concepts of fair sharing using their experiences; (2) students can interpret the fair share of a whole through partitioning activities; (3) students can use fractions as a unit of measurement and practice their skills in iterating; (4) students can apply the concept of fractions as a function of a quantity and develop unitizing skills; (5) students can develop multiplicative reasoning. Objectives (1) and (2) relate to how to introduce fractions and construct the concept of partitioning by linking their experiences when learning division, then the partitioning concept is adopted to be applied for purposes (3) and (4). Then each of the sub-objectives is simultaneously used to apply the ratio concept (5). As such, a series of sub-goals directs to promote proportional reasoning.

METHOD

Design research was applied to reach the objective of the study. It includes three phases: (1) preliminary teaching experiment, (2) teaching experiment, and (3) retrospective analysis (Gravemeijer, 2004; Gravemeijer & Cobb, 2006). However, to focus on the research objectives effectively and efficiently, this article focuses on reporting the revised HLT.

Participant

The participants in this study were 30 fourth-grade students at one public elementary school in Jakarta City. These 30 students were involved in the teaching experiment stage. They consisted of 17 girls and 13 boys with an average age range of 9 - 10.

The participants learned straightforward fractions (e.g., 1/2, 1/3, ¼) at the second grade. In the third grade, they continued developing that concept and then learned the addition and subtraction of fractions in the fourth grade. Nevertheless, according to their teacher, the participants have a weak understanding of the concept of fractions. It may hamper them from dealing with applying the concept of fractional operations in the subsequent grades.

Intervention

Researchers conducted a preliminary teaching experiment based on a hypothetical learning trajectory (HLT) in the first cycle. Based on the first cycle in five teaching sessions, the HLT 2 was generated as the refinement of the first HLT. HLT 2 was then implemented in the teaching experiment classroom and used as a guide for teaching practices.

The teaching experiment was conducted in a real classroom that consists of thirty 4th grade students. The conducted nine-stages learning activities are pretest, interview after pretest, learning
1 (fractions as quotients), learning 2 (fractions as part-whole), learning 3 (fractions as measures), learning 4 (fractions as operators), learning 5 (fractions as ratios), posttest, and interview after posttest.

Data Collection and Analysis

First, this phase formulates conjectures about what happened and examines it using the available data. Second, this phase formulates conjectures as to why this happened and then examines it. In formulating and testing conjectures, all collected data is considered, especially the transcription of whole-class discussion and students' work. The retrospective analysis phase was carried out based on all collected data during the teaching experiment. In this phase, researchers employ HLT as a guide in answering research questions. Subsequently, researchers develop local instructional theories and apply them to more general research topics.

The data was collected through tests, interviews, observations, and documentation. Before and after the intervention, we conducted tests and interviews to determine the improvement of HLT. In this study, an unstructured interview was carried out, and during observation, the role of researchers is as a participant-observer. All events that occurred were documented and recorded by two cameras. The first camera is static (static camera), which is intended to record all activities in the classroom, and the rest is dynamic (dynamic camera) to record particular activities during classroom discussions—professional videographers involved in this process.

Concerning internal validity in this study, the retrospective analysis is applied by researchers to interpret data obtained from interviews, observations, and documentation. The instrument used was examined before the implementation phase, then analysis was carried out to test the assumptions about students' thinking and learning processes. Besides, important events in each part of the learning process were recorded through video recording to obtain meaningful contexts. Furthermore, external validity focuses on the results obtained in different situations, guided by the question of how particular elements of the results obtained will apply to other situations. In this study, researchers try to improve reliability by discussing necessary findings in design experiments among researchers and improving the findings simultaneously.

RESULTS AND DISCUSSION

Teaching experiment 1: Transitioning from the whole number to fractions by student experiences
We begin this trajectory by introducing the meaning of fractions as division. We refer to the results of previous studies which show that students are weak in the concept of "equal share" (Purnomo et al., 2014, 2019) in which it causes learning obstacles to the application of other fraction concepts, such as sorting and comparing fractions and also fraction operations. It is pretty challenging because, in general, the part-whole approach is first used to introduce the concept of fractions. Moreover, in Indonesia, elementary school mathematics textbooks emphasize this part-whole approach to introducing the concept of fractions rather than other subconstructs (Purnomo, 2015; Rahmawati et al., 2020; Wijaya, 2017). Nonetheless, we consider our students' experiences when learning fractions before the part-whole concept using the whole number division as a foundation. Besides, a smooth transition between numbers and fractions becomes critical to develop proportional reasoning abilities (Im & Jitendra, 2020).

At learning 1, the teacher employs the context of the division of three crackers to six people through pieces of paper provided. Different strategies are found based on student group discussions in determining pieces in which they indicate different pieces and equal sizes. Some students divide the first crackers into six parts, followed by the second and third crackers, and then distribute them so that each person gets three-one-sixth. Meanwhile, some of them divide three crackers into six people by splitting each cracker into two parts so that each person gets half. The teacher then asks one group to show that division is to produce equal pieces (see Figure 3).

Figure 3. An example of group work in showing two different strategies for distributing crackers

Figure 3 shows that students use two strategies, in which the left part indicates distributive partitioning, while the right one indicates non-formal partitioning (halving). Empson (see in Jones, 2012; Purnomo, 2015) stated that children already know halves and do problems involving
repeated halving without formal instruction. This partitioning activity simultaneously might develop students’ iterating ability and the sense of equivalent fractions.

Furthermore, students are engaged with complex problems and guided with three worksheets adapted activities suggested by Lamon (2012). The first worksheet consists of four problems that ask students to work individually and carry out a fair sharing of a certain quantity to a number of people by sticking and or illustrating the pieces in the answer column. The activity aims to develop partitioning skills. The second worksheet includes five questions that are discussed in groups. It requires students to work together in determining the way how to share by sketching the pictures. Finally, the third worksheet includes three descriptive questions that must be answered individually by students. Students are asked to think of alternatives in dividing cakes into unequal but congruent forms on this worksheet. Figure 4 is a sample of student work on the first worksheet.

Figure 4. A sample of children's work for the first worksheet

*Translation:*

*Task 1*: What are the steps to share five cakes fairly with four friends? Explain with illustrations!
Figure 4 illustrates the preserved-pieces partitioning strategy (Lamon, 1996), which shows students' strategy in distributing five pieces of bread by distributing them one by one to get one piece of bread and then dividing one remaining bread into four people. Hence, students get one piece and $\frac{1}{4}$ pieces. Furthermore, the activity on this worksheet guides students to partition according to their experience so that their answers vary. An idea appears to divide each bread into four at once and then distributed—each person gets five a quarter—known as distributive partitioning. Some are cutting it half by half, often referred to as halving—each person gets two half one quarter. Thus, in addition to developing partitioning skills, this activity helps lay the foundation for learning equivalent fractions, addition, and division of fractions in the future.

**Teaching experiment 2: Partitioning activity**

In learning 2, the teacher delivers students to learn the meaning of fractions as part-whole. The teacher starts by giving a part-whole case using a hidden grid flat shape (the piece looks unequal), as shown in Fig. 5. This problem is given since they have learned the concept of simple fractions in the previous class. It is essential to anticipate the child's variety of thinking about fractions as part-whole and develop children's thinking habits and sensitivity to fractions.

![Partitioning activity images](image)

Figure 5. The problem of the fractional part is not the same as the invisible line

*R*: What is the fractional value in figure number 1, S1?
S1: One-fourth

R: Then who knows what fraction is this (no. 2)? (All students are silent)

What if this one (no. 3) (all students are still silent)? Is it hard?

R: Now pay attention! If I draw lines, can you read the fractions after this? Try No. 2. What is the fractional value?

S2: four sixth

R: ok, the third one?

S3: two sixths

R: correct, last numbers 4 and 5

S4: one-sixth

S5: four-sixth

R: It becomes easy. Why?

S4 & S5: Because there is a hint line

Most students have difficulty solving the problem as the image of fractions does not match their previous mental image, in which the part and the whole structure are regular. For this reason, the teacher provides a stimulus by asking students to match the group of shapes next to them and asking students to conclude. Through the question and answer process and adding helplines that divide the flat shape of the same size, students can determine the fractional value for the colored part. This activity can certainly be used when encountering the same problem, namely marking it with a hint line or their mental image.

The next activity was carried out by developing their understanding of the fractional part’s concept with a hidden grid. There are four worksheets to accommodate these activities. The first worksheet includes one question that asks students individually to take a propositional attitude and explain their reasons for the six shaded parts of the structure. The second worksheet involves arranging and matching the available shapes (i.e., equilateral triangles, rhombus, parallelogram, trapezoid). It is carried out to form the required fractions of a hat-shaped shape and place it on a paper-lined
with triangular grids. This problem adopts the activities suggested by Roddick and Silvas-Centeno (2007). The third worksheet consists of three description questions that are relevant to their daily experiences. The fourth worksheet contains five-word problems that students must answer individually in with each item requires students to use reasoning and estimation. The following is sample student work in learning 2.

![Figure 6: A sample of group work in stringing and matching activities (block patterned)](image)

**Translation:**

*How do you arrange the geometric shapes given (triangles, rhombus, parallelogram, and trapezoid) to become a figure of the following fractions: \( \frac{1}{2}, 1\frac{1}{2}, \frac{2}{3}, 1\frac{2}{3}, \frac{1}{3}, 2/3, \text{and } 4/3 \text{ of the hat}?*

Figure 6 shows that students use patterned blocks to determine the correct piece and whole. When they are asked to show \( \frac{1}{2} \) of the hat, they have to find the whole shape of the chosen shape. When the triangle is chosen as a unit, they find six triangles in the hat, so they will pair three of them to represent \( \frac{1}{2} \). There is also a pair of one trapezoid isosceles for \( \frac{1}{2} \) because one trapezoid is equivalent to 3 triangles. Various student representations were obtained when completing this activity.

In this activity, students are capable of determining the whole of the part from the fractional part and developing their partitioning skills to determine the fractional part of the whole of the part. In other words, students focus on the part of the fractions and the relationship to the whole. This ability can be called *unitizing* (Lamon, 1996). Unitizing is the mental ability to assign units...
different from the size of a given quantity (Lamon, 1996, 2012; Purnomo, 2015). This activity also bridges them to learn equivalent fractions (Purnomo, 2015).

**Teaching experiment 3: Iterating activity**

We introduce the concept of fractions as measures to develop students’ sensitivity regarding fractions as a quantity (Watanabe, 2006), developing proficiency in additive operations on fractions (Charalambous & Pitta-Pantazi, 2007), and developing their partitioning and iterating skills that are useful for multiplicative reasoning (Singh, 2000). Therefore, this concept is often introduced with the use of number lines or other measurement devices.

In the beginning, students are given a number line, and the teacher asks students to partition until they can mention the fractional value at the point marked on the number line. Under previous predictions, that students will have difficulty in partitioning gave number lines. This is supported by several researchers (Charalambous & Pitta-Pantazi, 2007; Purnomo et al., 2014) who found that students encounter difficulty when working with the number lines such as determining the location or position of fractions, calculating partition marks rather than intervals, determining the fraction unit, and determining the fraction between two fractions in a number line. Therefore, we ask students to use a ruler and partition it into four sections following the mark on the ruler. This method is enough to help students understand how many fractions are requested (i.e., 1/4).

![Figure 7. Students use iteration activities](image-url)
To understand how to name the fraction value on the number line, the teacher gives a probing question by asking students to partition it into a different number of 8 and asking how many fractions were marked beforehand. Figure 7 shows that students determine 3/8 and from a number line. Students do iterate that there are 8 of 1/8s units in 1 unit and sort the positions of 1/8, 2/8, 3/8, and $1 \frac{1}{8}$ simultaneously.

We further explore the concept of fractions as measures using two design tasks. The first task requires students to measure a specific fraction starting from the fraction unit using origami paper media. Activities on this task are intended to use various media, refine paper folding activities, and use number lines in the first cycle. Next, the second task requires students to determine a known fraction on the number line and wants students to determine the fraction between two numbers. Here is one sample of student answers when working on the first task.

![Figure 8. A sample of answers from measuring activities](image.png)

Figure 8 shows activities that involve partitioning and iterating skills (measurement). For example, in case number 2, students prepare five origami papers (partitioning) to measure 3 of 1/5s to get 3/5 (iterating) and case number 3, dividing six origami papers and counting 5 out of 1/6s to get 5/6. This activity becomes essential because it requires children to focus on partitioning skills requiring them to focus on parts (intervals) of origami paper rather than partition markings such
as number lines. This activity is helpful to avoid students' misconceptions about measurement (c.f. Wijaya, 2008). This activity also involves the iterating activity, copying the fractional unit to obtain the larger portion requested. The partitioning and iterating activities involved in learning fractions as measures become important in mathematics classrooms. It helps students develop the concept of fractions and develop multiplicative reasoning skills, which are the basis of proportional reasoning.

**Teaching experiment 4: Unitizing activity**

In learning 4, the teacher uses the button to introduce the concept of fractions as operators. Fractions as operators have been implemented by students in their daily lives even though they have not yet learned fractions formally. Students are familiar with the sentence: ask for half? One quarter? Or a fraction of this? These informal questions are adopted to raise context-based issues relating to fractions as operators. The teacher asks students to determine a fraction of the 24 buttons that each student has brought, which is as follows.

1. What is half of the 24 buttons?
2. What is one-third of the 24 buttons?
3. What is a quarter of 24 buttons?

![Figure 9. Activity to group 24 buttons based on specific fractions](image)

Figure 9 shows how a child groups 24 buttons into specific fractions according to the teacher's instructions. When the instructions ask them to look for 1/2 of the 24 buttons, with the teacher's
direction, they group them into two equal parts so that 1 of 2 groups of buttons is 12. Similarly, when the instruction requires 1/4 out of 24, they grouped them into four equal parts. With this activity, children can use their partitioning and unitizing skills. Partitioning skills occur when requiring students to make the same number of parts as a whole, while unitizing is seen when students do 24 group buttons into 2 (12 buttons), 3 (8 buttons), and 4 (6 buttons).

Furthermore, the teacher also develops students' sensitivity to fractions as operators using number machines like Figure 10. This activity adopts the idea of Lamon (2012).

![Figure 10. Machine numbers for fractions as operators](image)

In Figure 10, the teacher demonstrates a number machine to determine the operator that converts input 8 into output 4. The transformation process of input and output is carried out as the previous one. However, this machine draws students closer to the function of fractions as operators, namely transforming from a certain quantity (input) and produce another quantity (output).

**Teaching experiment 5: Transitioning additive to multiplicative reasoning**

Before understanding fractions as ratios, the teacher introduces multiplicative relationships using the definition of ratio. The ratio is a multiplicative comparison of two quantities or sizes, written with \( a : b \) (Purnomo, 2015). Multiplicative reasoning requires students to differentiate between multiplicative relationships (multiplication/division) and additive relationships (addition/subtraction) between two quantities. This context is then used to ask students to identify the relationship between nine green and three orange balls, as illustrated in Figure 11.
Figure 11. The teacher makes open questions related to the relationship between the number of orange and green balls.

Figure 11 shows one activity identifying the relationship between the number of orange and green balls. An example of student answers is "the number of green balls is more than the number of orange balls, the number of green and orange balls is 12, the number of green balls is six more balls than orange balls, the number of green balls is three times the number of orange balls, the number of balls orange quarter of the whole ball". These responses can be expressed as $x > y; x + y = 12; x = y + 6; x = 3 \times y; x = \frac{1}{4}z$ respectively, so that it can be used to differentiate between additive or multiplicative relationships. This mathematical statement which includes multiplication and division relations (known as multiplicative relations, i.e. $x = 3 \times y$ and $x = \frac{1}{4}z$) is what is then introduced as a ratio. It is essential to bridge the transition of additive to multiplicative reasoning. Through this activity, students also learn about some variations in ratios that are appropriate and non-ratios as one of the essential components in proportional reasoning (Lamon, 2012; Langrall & Swafford, 2000; Purnomo, 2015). Furthermore, the teacher provides reinforcement that the ratios stated previously have differences. The first case is the type of ratio as a part-whole relationship — the number of orange balls a quarter of all balls or $(x = \frac{1}{4}z)$ or 1 : 4 — which is also known as a fraction as previously learned. The second case is the type of ratio as part relationships — the number of green balls three times the number of orange balls or $x = 3 \times y$ or 3:1 — which is the ratio but not the fraction. The same activity was continued with varying numbers and colors of balls to develop their sensitivity to the meaning of fractions as ratios.
Armed with their understanding of value fractions, order fractions, and fraction comparisons, we began to introduce the concepts of propositions and non-propositions, namely by presenting pairs of fractions that were equivalent to 3/4 and 5/6. Students were asked to see whether they were the same or different. However, all students were silent and could not answer the questions because they were confused. The teacher gives a demonstration using the media bar on the board and compares it.

![Figure 12. The teacher uses the media bar to do iterating](image)

Figure 12 shows that the teacher performs iterating bars of multiples of four and six to get the same unity. The top bar is obtained by iterating three times, while the bottom bar is twice. After getting the same unity, the teacher compares 3/4 and 5/6 by pointing to 3 copies of the 3/4 top bar and two copies of the 5/6 bottom bar. With shading, that comparison is equivalent to 9/12 and 10/12. This activity indicates that partitioning, iterating, and unitizing are helpful in completing fraction comparisons, developing their multiplicative reasoning, and establishing their proportional reasoning.

**CONCLUSIONS**

Design research is carried out to provide a general overview for teachers about instructional sequences in student learning and learning processes. This study's findings indicate that the developed learning trajectory can lay essential foundations in assisting fractions concept
development. There are five main learning activities as a trajectory to promote it, namely (1) starting with introducing fractions as the division of whole numbers; in fair sharing, the obtained pieces do not have to be congruent; (2) exploring and developing partitioning activities on the concept of fractions as part-whole; (3) using the concept of partitioning to get a relative unit of measure from fractions and to transition from discrete concepts to continuous concepts; consider fractions as a systematic measurement related to the use of a number line or other measurement device; (4) employing partitioning capabilities to unitize and operate numbers; (5) using partitioning, unitizing, and carrying out operations as the basis for multiplicative relationships and proportional reasoning.

There are several key features in our research findings in laying the foundation for developing the concept of fractions, namely (1) the importance of the transition between the whole numbers and fractions so that using the prior experience is the best alternative. In the context of our study, namely by using whole number division to introduce the concept of fractions; (2) the importance of the concept of partitioning for the five subconstructs, for example, the students learned the quotient sub-construct while finding the fair share of 3 crackers among six people. They accomplished this by an informal halving technique (informal operator) while others divide each successive cracker into six parts and then distribute. Thus, this second reasoning process involved part-whole partitioning followed by coordination – distribution and was thus as a learning method was a good application of part-whole partitioning within the quotient subconstruct; (3) the importance of bridging the transition of additive and multiplicative reasoning, so we use the ways that students identify these two traits with independent and differentiating investigations and use them to conceptualize the concept of ratios.

Following the research design's objectives, the implications of this study should be considered by teachers. Teachers are expected to go out of their routines and explore the concept of fractional parts that are not focused only on fractions as part-whole relationships, even when introducing it to students for the first time. This study suggests starting with fractions as division, which they have learned before learning fractions. We focus on this, even though other subconstructs are also very attached to their daily lives. We also find that the findings of this research help prepare a mathematics curriculum based on research-based learning trajectories rather than merely adapting or copying curricula in other advanced places. It is important because each student and teacher must be exposed to teaching and learning situation which fit their characteristics and their respective environments.
Our study's limitation is the time constraint in which we conducted only five preliminary teaching and five teaching experiments. It has implications for the development of each subconstructs, which needs to explore more free play activities. Therefore, other researchers need to manage this and discuss it with responsible parties. We also delimit the critical aspects of proportional reasoning developed through subconstructs of fractions so that other researchers can use this research to enrich cases directly related to the problem of proportionality or others. We also suggest a need for substantial research to determine the order of mathematical content for each grade level. As it might be a complicated effort, thus researchers and teachers need to work collaboratively and guarantee research-based practice based on students' context.

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The Calculus Concept Inventory Applied to the Case of Large Groups of Differential Calculus in the Context of the Program “Ser Pilo Paga” in Colombia.

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Abstract: The Calculus Concept Inventory (CCI), Epstein (2013) aims to test the understanding of calculus ideas, rather than ability to perform calculations. In this paper the CCI is used to measure the effect of the undergraduate calculus cohort over the understanding of calculus in a heterogeneous population including recipients of the program Ser Pilo Paga (Pilos). There is a global positive gain of 0.10 (3), a weak correlation between gain and the percentage of Pilos, a negative correlation between initial score and the gain, and no correlation between class size and gain. The values hopefully would provide a baseline for comparing future interventions on the teaching of calculus.

The prime goal of teaching calculus for non math-major students is to achieve an understanding of the mathematical concepts and their relations. In this paper, I aim to gauge the outcome of the education process in different groups of students that took a Differential Calculus class at the Universidad de Bogotá Jorge Tadeo Lozano during the second semester of 2016. I have characterized the results of students within different subgroups: by gender, recipients of “ser pilo paga” (a state-sponsored program aimed at low-income students), session size, and initial score. The results contribute to the understanding of the differences between Pilos (recipients of ser pilo paga") and the general population of this university.

Firstly, a definition of the understanding of first-year calculus is needed. (Sofronas, et al., 2011) interviewed 24 experts and book authors and concluded with a set of goals and sub-goals, organized in a framework (a hierarchically organized list). For each goal of the framework, the
authors reported the percentage of experts who think it is a key element of understanding. The basic general goals in the survey are: “(a) mastery of the fundamental concepts and/or skills of the first-year calculus, (b) construction of connections and relationships between and among concepts and skills, (c) the ability to use the ideas of the first-year calculus, and (d) a deep sense of the context and purpose of the calculus” (Sofronas, et al., 2011). Let me point out to the fact that there was consensus only on one goal: “Mastery of Fundamental Concepts and/or Skills”, thus there is no agreement among experts on what constitutes understanding of mathematics.

Furthermore, what has been considered “fundamental” has changed over time. Historically, math education has swung back and forth between two extremes. On one end, there is have traditional education, also called procedural teaching, which puts a strong focus on the teaching of basic skills. Their proponents argue that the acquisition of those skills lowers the cognitive load needed to understand higher concepts, as well as provides basic facts that are important in the construction of higher-order concepts and their interpretation (NCTM, 2021). On the other end, there are proponents of high order understanding (Österman & Bråting, 2019). They claim that high-level skills can and should be taught, thus the focus of the education effort should lie in, for instance, problem-solving and creative thinking. There is a large divide between the two communities, for instance, Berry and Nyman (2003), report results of interviews highlight a better understanding of the algebraic symbolic view of calculus over the graphical representation (presumably a traditionalist approach); Habre and Abboud (2006) called the traditional approach to calculus teaching, as “teaching techniques for solving drill problems”, (p. 57); while comparing to a reformed calculus class, which emphasizes visualization (Habre & Abboud, 2006).

A particular goal of the calculus reform movement was to give the students fluency in the use of multiple representations. Since computers can perform the procedural, pencil-and-paper algorithmic techniques (called pejoratively “symbol pushing”), mathematics should involve more visual work and real work scenarios (Zazkis, 2013). In contrast conceptual knowledge is described as including graph interpretation and creation skills, knowledge of various representations and how to translate between them, ability to derive procedures from basic principles, ability to tackle novel problems, physical interpretations of functions/graphs that describe motion, and the ability to translate word problems into calculus equations (Zazkis, 2013). Contemporary calculus textbooks like Stewart Calculus, used in this study include several exercises that aim to improve the understanding of representations.

Now, is there really a split between mastery of fundamental concepts or skills and a deep sense of context and purpose of calculus? It is common in discussions among colleagues or the literature-
to see it referred to as a dichotomy: a class could focus on either of them. I adhere to the view that there is no such dichotomy. On the contrary, there is a reinforcement loop between the acquisition of skills and the understanding of the purpose of a calculation. Thus, understanding of higher order concepts both requires “the mastery of fundamentals concepts and skills” and helps to achieve it. (Wu, 1999, p. 3)

Is in this context that Concept Inventories were created. They are specifically designed to test the comprehension of the conceptual base of a given subject. In order to solve the questions, students have to be able to apply the principles to simple but interesting situations; where usual calculations and algorithms are of little or no use (Epstein, 2007). The first Concept Inventory was the Force Concept Inventory (FCI), by Hestenes, et al., (1992). Here I use the Calculus Concept Inventory (CCI), (Epstein, 2006, 2007, 2013), in order to assess the students’ conceptual understanding of basic Differential Calculus. The CCI has been used in different contexts to measure the understanding of calculus concepts (Rhea, 2008). One of the claims by Epstein and others about calculus understanding, in which the CCI has been used as an argument is this: the single most important factor increasing the understanding of calculus concepts is the use of the instructional style called Interactive Engagement (IE) (Epstein, 2013, Thomas & Lozano, 2013). Broadly speaking, an IE oriented teaching style looks for ways to have the students actively participating in the classroom activities; compared to the traditional lecture in which they are only recipients of the knowledge that is being given to them.

Within this framework, I wanted to test the level of understanding of calculus concepts by students of Differential Calculus at the Universidad de Bogotá Jorge Tadeo Lozano (UTADEO). The population consists of undergraduates, some of whom (about 1/3) come from a state-sponsored program aimed at low-income students. Another particularity of the present study is the use of a large session (about 100 students) to present the theory of mathematics; previously (until 2015) most calculus courses consisted in small groups (less than 30 students). First, I report how this particular cohort of students and this institution compared to other institutions who have taken the CCI. Secondly, I use the results of the CCI as a way to characterize the different subgroups of students.

Regarding the structure of this article, in the section Materials and Methods I characterize the student population and explain the methodology. In the section Results I discuss the overall results and findings. Finally, in the section Conclusion I have included the closing remarks and concussions.
MATERIALS AND METHODS

Population
This study was conducted with students of Differential Calculus at the UTADEO. This is a Colombian university, located in Bogotá. There are some particularities regarding the population:

- The Colombian government has a test for high school students, Saber 11 (ICFES, 2014). Since the year 2015, the Colombian government started a scholarship program “Ser Pilo Paga”, aimed at those students who do well in this test (the specific threshold varied during the existence of the program), and cannot afford the cost higher education. Those students are referred to as Pilos. Some of the students in this study are Pilos. (ICETEX - MEN, 2015)

- While most Colombian universities use either the state test or internal knowledge testing in order to filter admission, the UTADEO does not. The point in case is to give the opportunity of higher education even to students whose previous background has not given them enough opportunities.

- The first session of the week is given by full-time professors. They present the concepts to a large class (about 100 students). In class, grading is done by using clickers. Both the large and the small sessions happen at the same time of the day, at different days of the week. There were four of those large classes, given by three different professors.

- In the second session, students participate in smaller groups (about 25 students), led by a part time teacher; there they solve questions, exercises, and problems in a hands-on approach. I obtained averaged gain for the small session groups, as well as the percentage of Pilos in the small sessions. There were 20 small sessions, led by 12 part time teachers.

- The book used throughout the class is “Single Variable Calculus: Early Transcendentals”, by James Stewart (Stewart, 2015). As with most contemporary calculus books, this textbook has been influenced by the reform movement. Therefore, it includes, for instance, the graphical, analytical, and numerical treatment of functions.

- The first CCI test was administered during the first semester of 2016 (January through May).
Table 1 shows the characterization of the population of students in Differential Calculus. Causes of the difference in population between those in the CCI study and the universe of the Differential Calculus population (521 students) include students who did not attend class the day of either the first or the second application of the CCI test, students who withdrew the class, and students who did not fill the gender information in the questionnaire. I did include in the CCI study students who did not fill the fields related to the prior Differential Calculus knowledge, gender, or participation in the Ser Pilo Paga program.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Students who signed up for Differential Calculus</td>
<td>521</td>
</tr>
<tr>
<td>Percentage of signed up who did not cancel their enrollment</td>
<td>90 %</td>
</tr>
<tr>
<td>Percentage of students who did not cancel their enrollment and later on approved</td>
<td>64 %</td>
</tr>
<tr>
<td>Percentage of students who did not cancel their enrollment but later on attendance</td>
<td>3 %</td>
</tr>
<tr>
<td>Percentage of students who did not cancel their enrollment and later on failed by grade</td>
<td>33 %</td>
</tr>
</tbody>
</table>

Table 1: Characterization of the Differential Calculus Students

Table 2 shows the characterization of the population in the CCI test. It is worth mentioning that Pilos make up 1/3 of it, and only 31 (10%) of the students claimed to have studied calculus before.

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<th>Quantity</th>
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<tbody>
<tr>
<td>Number of students</td>
<td>305</td>
</tr>
<tr>
<td>Males</td>
<td>158</td>
</tr>
<tr>
<td>Females</td>
<td>147</td>
</tr>
<tr>
<td>Pilos</td>
<td>105</td>
</tr>
</tbody>
</table>
Students who have studied calculus 31
Students who have studied pre-calculus 49

Table 2. Characterization of the Students in this study

Fig. 1 is a histogram of the distribution of ages of the students within the study. It peaks at 18 years of age. Most of the population (254 students) is younger than 22.

![Distribution of ages of students in the study](image)

**Figure 1. Distribution of ages of students in the study**

**The Test**

In words of Epstein (Epstein, 2006): “The calculus concept inventory (CCI) is a test of conceptual understanding (and only that -there is essentially no computation) of the most basic principles of Differential Calculus”. The 2006 version consists of 22 multiple options - single choice questions. In this study, I translated the original English text to Spanish.

In the context of the CCI, the improvement of the understanding of calculus by a group of students is measured by the normalized gain. Epstein, (Epstein, 2013), defined it as:

\[
\langle g \rangle = \frac{\mu_f - \mu_0}{s_{max} - \mu_0}
\]  

(1)
with $\mu_0$ the mean score of the class at the first test, $\mu_f$ the mean score of the class in the second test, and $s_{max}$ is the maximum possible value of the score (22 in this case). The gain measures: “the gain in the class’ performance (...) as a fraction of the maximum possible gain” (Epstein, 2013).

In the present work, I have calculated the gain for each of the 20 small sessions, as well as for some subgroups of students characterized by different characteristics: gender, Pilos, and initial score.

**RESULTS**

For each of the 305 students I have both the initial and final value of the score of the test. The gain defined by Equation 1 is interpreted as the improvement of the understanding of calculus by the whole population. To determine an uncertainty of the gain I interpreted the standard deviation of each of $\mu_0$ and $\mu_f$ as the uncertainty of each measurement. Therefore, the uncertainty of the gain is given by:

$$
\delta g = \frac{\mu_f - s_{max}}{(s_{max} - \mu_0)^2} \delta \mu_0 + \frac{\delta \mu_f}{s_{max} - \mu_0}
$$

where $\delta \mu$ is the standard deviation, $\delta g$ uncertainty in the gain.

This gives a gain of $g = 0.10 \pm 0.03$, which is clearly statistically significant gain. Now, since the gain relies in average values this gain could be the result of a few individuals making strong improvement, but this is not the case here as can be seen in Figure 2. At the left, in black, the distribution of pre-class test scores; at the right, shaded, the post-class test scores. It is clear that after the class the distribution of test values is shifted towards higher score values. Thus, as measured by the CCI, I conclude that taking a one semester class of Differential Calculus resulted in a clear improvement in understanding.
Figure 2. Distribution of raw scores, both for pre- and post-test

Since the gain of the whole population only gives a global characterization, I proceeded to calculate the gain that different sub-groups have had on the test. This was applied as an exploration of the data that could guide the improvement of the teaching of calculus and could help us find where I should focus my efforts. I did not explicitly start from a hypothesis of these groups to have had an advantage, on the contrary, I wanted the data to show the inequities. The result of this process is summarized on Table 3. Every value in this table is statistically significant, as can be seen comparing the respective values and the uncertainty. Of these groups, Pilos have a higher gain 0.12, compared to 0.09 of no-Pilos. Males also improved more than females, with 0.11 compared to 0.09. The best session, session 6.2, scored 0.27, a very impressive value compared with the rest of the students. The worst session, at 0.05, did half of the improvement than the general population.
Students attended two sessions a week, and the second session was led by a part time teacher and held in smaller groups. Does the group size impact the understanding of calculus? From Figure 3, I concluded that there is no correlation between the second session group size and gain.
Figure 3. Normalized gain vs. complementary class size. The horizontal line corresponds to a linear regression giving a slope of 0.04, and p-value $5 \times 10^{-6}$. Therefore, one can say with significance that there is no correlation between second (small) session size and gain.

Furthermore, I wanted to find out whether the presence of Pilos in a group had an effect on the class performance. Indeed, as the percentage of Pilos increases, the gain tends to increase also. This is seen in Figure 4, in which I have plotted the normalized gain against the class percentage of Pilos in the second (small) session. My analysis yielded a positive correlation represented in a line with slope of $0.097 \pm 0.056$. Nonetheless, the p-value is 0.1, so the analysis is inconclusive.
Figure 4. Normalized gain vs. percentage of *Pilos*. The dashed line corresponds to linear regression analysis with a slope of 0.097; the p-value is 0.10.

In Figure 5, I have plotted the normalized gain against the initial class score. Since I am averaging over the population of students who obtained a given score, I have included error bars. The last two points (12 and 14) correspond to the gain of single students; therefore, the concept of gain hardly can be applied to them and is safe to disregard them. A decreasing trend for low scores can be seen in the data, in which having a higher initial score leads to a smaller gain. One hypothesis is that having a higher initial score implies a lower chance of improvement, so gain decreases.
Figure 5. Normalized gain vs. initial raw score: the better the initial score, the lower the gain. The dashed line corresponds to linear regression analysis with a slope of -0.038; the p-value is $5 \times 10^{-6}$, therefore the correlation among the variables is significant.

Another question of interest was, is there a relationship between the time of the day the lessons were held and the gain? Additionally, is there a relationship between the years of experience of the professor in charge of the large sessions and the gain? For the data, these two questions are related, as the most senior professor also taught the early sessions. The data is shown in Figure 6. Average normalized gains of the small session groups that were held at 7:00 is significantly higher than those at 9:00, 11:00, or 18:00. Now, since students choose their schedule, time of the lesson could be a proxy to other characteristics of the population. The effect, however, could also be attributed to the experience of the professor in charge of the large class, as she has over 30 years of experience against 10 years of experience for the full-time professors in charge of the other large groups.
CONCLUSION

I have applied CCI, a conceptual test of Differential Calculus, to 305 students of the subject, in Bogotá, who attend a class with both a large session (around 100 students in a lecture room) and a small session (between 14 and 25 students); each session is 2 hours, once a week. There is a positive gain of $0.10 \pm 0.03$ when taken over the whole population of the study, which indicates that there is a better understanding of basic concepts after taking the class. However, the gain is small, when compared to the reported data from University of Michigan (with an average of 0.35 (Epstein, 2013)).

Since students were split into 20 small sessions, and the percentage of Pilos (students belonging to a national excellence program) varied among small sessions, I have found evidence of a positive correlation between the number of Pilos and the normalized gain, however, the statistics is inconclusive. Having some Pilos in a classroom may have improved understanding of Differential Calculus concepts. I expect the results of the present report to be a useful baseline. Therefore, future changes to the curriculum or other teaching intervention can be compared against the values herein reported, and therefore, could provide useful information. Moreover, researchers in similar institutions can compare their results against ours.
There are open questions. I have obtained a negative linear correlation between the initial test score and the gain. This trend is different from what is usually stated in the literature: that there is no relationship between the initial score and the gain. It would be interesting to see if this is seen in different data sets, since the p-value of $5 \times 10^{-6}$ gives confidence in my analysis. A possible interpretation is that the students having a higher initial score diminishes the possibility of improvement. However, I have also been aware of the criticism of the CCI, according to the work of (Gleason, Thomas, et al., 2015) the CCI relies strongly on notation rather than on conceptual understanding relating multiple representations. I do not plan to address the shortcomings of the CCI in this work. Further studies could bring more light into this issue.

Finally, the effect of seniority of the teacher with the time of the day the class is taught cannot be disentangled, therefore it is impossible to conclude whether any of those variables has an impact on the outcome of the students. These are still an open questions that should be tackled in further research.

Limitations of the CCI

The main weakness of the study is its reliance on the CCI itself, which is not exempt of criticism. According to a factor analysis reported by (Gleason, Thomas, et al., 2015, Gleason, White, et al., 2015), the CCI can be used to explain one single factor, “overall knowledge of calculus content”; rather than distinguishing between three factors, functions, derivatives and “limits/ratios/the continuum”. Furthermore, the internal reliability of the CCI (a Cronbach alpha of 0.7) is below the standard (0.8). On his work on the subject, Gleason does not call for a dismissal of the CCI, rather he wants it to be improved to overcome these difficulties. Such modification of the CCI is beyond the scope of this paper, but the author would gladly collaborate on such endeavor.

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References


