Editorial from Bronislaw Czarnocha

One of the important changes MTRJ is going through is the diversity of submitted and published papers with respect to their national origins. Since we are delighted by the process, we decided to indicate the regions of origin of papers both in the List of Content as well as in the Editorial.

The first two papers of the current issue give, possibly the last look at the teaching during pandemic, informing, from West Papua, Indonesia about particularly difficult situation in rural areas of the country where Internet connection has been absent. The teachers, clearly essential workers, have been visiting homes of individual children and teaching in their homes. Obviously, they were spending 2-3 times more teaching hours than their regular load in addition to preparation of materials and homework grading. The authors propose Blending Learning as the proper path for West Papua.

On the other hand, the second paper by Haoyi Wang, a graduate student from Illinois, tells us what works and what does not while teaching online. Both papers are continuation of our discussion of the pandemic teaching from the Vol. 12 N 3.

The next four papers open wide discussion concerning different aspects of mathematics teachers’ knowledge. Breda et al. question the concept of improvement of learning. They find out that accordingly to newly prepared teachers’ improvement of learning is possible in the cognitive, ecological, and emotional aspects and, to a lesser extent, in the interactional, mediational and epistemic aspects. Their discussion is in the context of scientifically deciphering the meaning of didactics itself, and what is especially interesting, it is based on the data collection from three different countries: Spain, Chile and Ecuador. On the other hand, the work by Jay Aguilar from the University of Texas, Rio Grande Valley, looks at the didactics of modelling and finds it very suitable for teaching both high and low achievement students. It is a very optimistic and motivating result; it corresponds to the formulation of mathematical creativity of Aha! Moment as the theory of creativity for and of all students (Czarnocha and Baker, 2021). Aguilar’s work is complemented in this issue by Piñeiro et al., the work of colleagues from Chile and Spain, which investigates teachers’ preparation as the problem-solving instructors. They find that neither of the Problem-Solving models present in the Math Education profession does justice to the requirements of the problem-solving pedagogy. They ask an important question how teachers’ knowledge about PS is actually used in classroom practice – an excellent teaching-research question, in our opinion.
The issue closes with the analysis of the didactics of collaborative teaching penned by Amjad Ali et al. our colleagues from Pakistan. Collaborative teaching is very close to our hearts in the South Bronx, as it is one of the routes of facilitation classroom Aha!Moments. The authors used standard method of the experimental and control groups of the fifth grade students and were able to demonstrate that collaborative teaching results in statistically significant achievement improvement.

Bronisław Czarnocha
Chief Editor of MTRJ

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The Mathematics Instruction in Rural Area during the Pandemic Era: Problems and Solutions

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Abstract: The pandemic era had an enormous influence on teaching and learning activities in all regions of the world. For urban areas that generally already have a variety of adequate facilities and infrastructure, it still has an impact on their learning activities. However, this outbreak in rural areas with limitations in teaching and learning activities has its own story. Therefore, this study aims to identify the problems encountered and the solutions implemented by teachers, lecturers, and students in the implementation of mathematics learning during the COVID-19 pandemic in one of the rural areas in Indonesia, namely Manokwari, West Papua. Teachers, students, and lecturers were all purposefully selected as research subjects for the study, which was conducted using qualitative research techniques. Data was collected through structured interviews using the WhatsApp application, then analyzed to construct narratives, tables, and images. The results showed two main problems in implementing the online mathematics learning system in West Papua, namely accessibility to Information Communications Technology (ICT) equipment and the ability to use ICT equipment in carrying out mathematics learning online. The results also show that online mathematics learning is necessary to require government involvement in planning, implementing, and evaluating online mathematics learning systems. Yet, blended learning is a learning system that is suitable to be applied in West Papua during this pandemic situation.

INTRODUCTION

Coronavirus disease (COVID-19) is an infectious disease caused by a newly discovered coronavirus. The disease caused by Covid-19 has spread from Wuhan to all of China (Lipsitch et al., 2020). The virus that emerged in December 2019 has spread rapidly, with cases now confirmed in multiple countries, include Indonesia. WHO later declared the disease as a Pandemic.

As a Pandemic, this disease has spread to almost all countries globally, including Indonesia. President of the Republic of Indonesia, Ir. Joko Widodo, reported that the first Indonesian citizens infected with the virus were two people in Depok, West Java, on March 2, 2020. Since then, the number and distribution of infections have increased. Recorded until May 22, 2020, there have
been 395 regencies in all provinces in Indonesia. Distribution COVID-19 positive in Indonesia is presented in Figure 1.

![Figure 1: Distribution of Covid-19 cases in Indonesia](image1.png)

Figure 1 provides information that residents in all provinces and large islands in Indonesia have confirmed COVID-19. This information shows that all Indonesians, from East to West, from North to South, will suffer the consequences of this pandemic without exception. Furthermore, the number of people who confirmed positive for this virus, the number who died, recovered, and treated was presented in Figure 2.

![Figure 2: National trends of Covid-19 cases in Indonesia](image2.png)
In contrast, Figure 3 offers information about increasing the number of coronavirus sufferers every day. Based on Figure 2 and Figure 3, it appears that the government cannot predict the end of the outbreak of Covid-19 in Indonesia. This fact is contrary to previous research reports that conducted studies to indicate that this pandemic will end in Indonesia before June 2020 (Susanto, 2020; Nuraini et al., 2020; Dwiputra, 2020). However, based on these data, these predictions are unlikely to occur. This pandemic can continue until the end of 2020, maybe even in the next few years, if the government and Indonesia's people are not severe in handling it.

![Figure 3: Daily distribution of cases of Covid-19 in Indonesia](image)

On the other hand, this pandemic has influenced Indonesia aspects, including Education (Hamid, 2020; Nadeak, 2020). There are direct impacts on teachers (schools), students, and parents (Giles, Park, & Wang, 2019; Wargadinata et al., 2020). A face-to-face learning system that has been going on all of sudden has to be replaced immediately by online learning systems (Owston & York, 2018; Krishnamurthy, 2020). Furthermore, the instruction has shifted from teaching face-to-face to teach online due to the COVID-19 outbreak. The tools, models, and learning systems that have been applied and studied by teachers must be replaced with an online learning system (Farhan et al., 2019; Al Masarweh, 2019; Thongsri, Shen, & Bao, 2019; Al-Fraihat, Joy, & Sinclair, 2020). Therefore, teachers and students have to change and adapt to the learning models and tools to carry out the online learning process correctly.

There are several problems faced in some rural areas, including Manokwari, West Papua. The limited infrastructure of Information Communications Technology (ICT) for teachers and students is also a problem in the online learning system. The distribution of ICT facilities is another factor that hinders the implementation of online learning systems in West Papua. The online learning
system is limited. Instruction cannot be executed according to the curriculum. Teachers, including students and parents, have a large workload because they are unfamiliar with the online learning environment. Instruction does not take place in the same way as it does in the face-to-face study. Furthermore, teachers have given some assignments, and students have worked without explanation before. Thus, a representative solution is needed to resolve this problem. This is because the obstacle in rural areas has a unique character compared to urban communities.

In this paper, five main topics will be discussed. First, how teachers, lecturers, students, and parents in West Papua, especially in Manokwari, address the changes of the learning systems. Second, what learning model are used in Indonesian West Papua during the COVID-19 period, third, what students think about the learning system. Fourth, the effectiveness of the learning system. And last, various steps taken by the government, schools, teachers, parents, and students to improve the quality of mathematics instruction in West Papua during the Pandemic period. Problems and solutions will be explored and explained in the next section.

RESEARCH METHODS

The study was conducted using qualitative methods with a descriptive approach. In this study, research subjects describe their experiences and knowledge about research objects, teaching, and mathematics learning in the COVID-19 period. Data collection was carried out through structured interviews with teachers, lecturers, and students, using the WhatsApp (WA) application.

Sampling was done among mathematics teachers and lecturers from junior high school, senior high school, or university who were teaching mathematics before and during the pandemic. The selection of students also used the same criteria, namely, students at a particular level of education who studied mathematics before and during the pandemic. There were ten teachers, two lecturers, and nine students who are the subjects of this study.

Interviews were conducted using the WA application. This research critical question was: “Please Mr. / Mrs. share experiences and problems in mathematics instruction during COVID-19 period.” The item then continued to find out more in detail about the learning system chosen, the reasons for using the method, various obstacles encountered, the solutions implemented, and the suggestions for further improvement of the online learning system for mathematics instruction in Manokwari, West Papua.

In addition to the primary data, secondary data from the literature, especially about learning mathematics and COVID-19, was used in this study. The data was analyzed and presented in tables and narratives.
RESULT AND DISCUSSION

During the COVID-19 pandemic teachers, lecturers, and students in Manokwari use the internet-connected smartphones and laptops to learn. These ICT tools run the software, such as WhatsApp, Zoom Conference, Google Classroom, Video Tutorial, and e-learning. E-learning is an online learning system used to complement face-to-face learning systems at the University of Papua.

Teachers, lecturers, and students face several challenges when implementing an online learning system. Various solutions have been implemented by teachers, lecturers, and students so that mathematics learning can be delivered in West Papua during this situation, as discussed further.

Problems and Solutions of the Online Learning System in West Papua

The first problem encountered by teachers, lecturers, and students in West Papua in implementing the online learning system for mathematics instruction was distributing information and communication infrastructure. Some Manokwari locations are far from the internet tower, so they cannot access the internet signal. In some places, the internet signal cannot be obtained at any time except in the evenings or mornings.

The second problem is the accessibility of information and communication technology (ICT) devices, smartphones, and laptops. Not all students (and parents) have smartphones that can be used to support online learning activities. The devices are limited and are used interchangeably, especially for parents who have more than one child attending school or college. Besides, some students have ICT tools and cannot access the Internet because it requires expensive financing.

To overcome these problems, the teachers carry out mathematics instruction in students' homes. However, the instruction outside of the school is an additional burden for the teacher. The teacher has to go to every student’s house because there is a ban on gathering. Learning activities for 80 - 90 minutes at school, the teacher must do about 180 - 240 minutes when visiting students’ homes. Teachers visits were becoming more frequent if students were not home during the session. Therefore, during the visit, the teachers provided well-prepared instructional materials. When using these instructional materials, students were expected to comprehend the content well. Yet, if students do not understand it, the teacher can go over it in the next visit.

The summary of teachers activities to overcome students difficulties accessing the Internet and ICT equipment are presented in Figure 4. It shows that teachers, even school principals, come to the homes of students who do not have access to the online learning system. The teacher visits students in their families to explain the subject matter, provide lesson material, deliver test material, and monitor the midterm and final exam. However, the activity of these teachers is a short-term solution. In the future, the answer is an improper step. The government needs to solve this problem to implement the online learning system in West Papua.
Especially for the lecturers at the University of Papua, ICT development has been in their learning activities. Lecturers have used several Online Learning Systems (OLS), such as WhatsApp, Zoom conference, Google Classroom, tutorial video, and e-learning. However, in practice, the OLS was not explicitly designed by the lecturer to be used separately from conventional lectures that prioritize face-to-face learning (offline learning). Lecturers use OLS and face-to-face learning alternately in their learning activities. In this case, OLS is positioned as support in their learning activities.

The next major issue is how to use ICT devices as a learning tool. Not all teachers, lecturers, and students are accustomed to utilizing this technological device in the online learning system. They are used to using smartphones and are limited to sending and receiving messages, especially using the WA application. Librero et al. (2007) explain that the cellular phone is not designed to be used in education, but it can be used as a learning tool. Teachers have to explore mobile phones' potential as a crucial device in the educational systems of developing countries.

To implement online learning with applications WA, mathematics teachers establish WhatsApp Group (WAG). The use of WAG is familiar for teachers and students in West Papua. They used to use WAG in their daily activities. This situation is in line with Sutikno et al. (2016)’s research results, which states that WA is the best apps for instant messaging. However, there is a difference between using a smartphone for the learning process and daily activities. Joo and Sang (2013) state that there are two types of smartphone usage: ritualized and instrumental. Ritualized media use is more frequent and used more for diversionary reasons. On the other hand, the practical application refers to a more goal-oriented use of media content to gratify "informational needs or motives."

Consequently, some problems arise during the implementation of online learning using WAG. Interaction between teachers and students is not going on well in the execution of mathematics instruction. The teacher asks students to learn the subject matter by referring to the textbooks and
student worksheets to complete the examples and then solve the exercise questions. Learning activities of teachers and students apply the online learning system shown in Figure 5.

![Figure 5: Teachers and students activities during online mathematics instruction](image)

Furthermore, although students have been learned from textbooks and student worksheet activities, those who do not understand the subject matter usually ask for both parents' explanations. Unfortunately, not all parents have the competency and opportunity to assist students in comprehending the subject matter. To answer the questions given, students then ask the answers to their friends. There is a tendency for students to answer the questions correctly without understanding the problem.

In an online learning system, the typical interaction occurs in asynchronous, text-based discussion forums. Teachers and students post messages and respond to other people's postings, resulting in a threaded discussion. In these discussions, if a teacher or learner does not display or is delayed in responding to another's post, the absence of communication comes across as silence (Xin & Feenberg, 2007; Duran 2020).

To solve this problem, some teachers asked students to create a video that shows how to resolve a particular issue. But in general, the teachers ask for students to work on the problems and then score without giving feedback to students. The use of video to ensure students' understanding of the subject matter, provided in Figure 6. It shows that students explain the stages to solve a mathematical problem. Students' ability to demonstrate these stages in detail, orderly and correct indicates their mastery of the learning material. The use of the video is an effort to increase interaction between teachers and students in learning. Teachers need to make more innovative approaches to achieve the learning objectives of mathematics instruction.
The Future of Mathematics Instruction in Online Learning System

Online mathematics instruction is necessary throughout the world, including in Indonesia and West Papua, especially in the COVID-19 Pandemic. The government must address multiple problems in the implementation of online math learning in West Papua. They should quickly provide solutions to overcome the issues.

The first step that needs to be done by the government is to organize the ICT tools for an online learning system. The government has to provide the infrastructure of telecommunication technology to support mobile and internet networks. ICT tools have an essential role in education (Ariyanti & Santoso, 2020). Furthermore, Zhang and Cristol (2019) stated that ICT has been used in higher education for many years. They provide reasonable solutions for Instruction and make Learning available anywhere and anytime.

ICT devices should be accessible to all stakeholders, teachers, lecturers, and students. They should be able to access the internet anywhere and anytime, especially at home. Furthermore, Whelan (2008) shows that government support is one of the essential development factors to improve access to ICT in The South Pacific. The South Pacific is a region with some similarities with the characteristics of West Papua's province in Indonesia.

Farley and Song (2019) explain that Indonesia has high mobile penetration levels but relatively low broadband internet and computer penetration levels. Broadband internet penetration is restricted due to poor infrastructure. On the other hand, on May 16, 2011, the United Nations stated that access to the internet was a human right. That statement has implications for governments in providing internet infrastructure (La Rue, 2011).

The second factor of access to the internet in West Papua relates to affordability. The cost of buying a phone, a sim card, and any upfront fees associated with holding a mobile phone can account for a large proportion of a person's income (Jeroschewski et al., 2013). This corresponded to the students' statements not to access the internet in West Papua. Therefore, the Indonesian government should provide subsidies to overcome this problem. The government can give open textbooks on this issue. It can be done by delivering free books to support online learning (Pitt et al., 2020).
Therefore, the Indonesian government should conduct a study before acting to resolve these problems. The review should involve all stakeholders, including teachers and parents. The study also needs to be done in all aspects, including economic issues. So (2012) states that the Indonesian government must study accessibility, connectivity, and affordability of mobile devices, especially in West Papua. Furthermore, the Indonesian government also needs to establish the National Standards for Distance Education (or online learning system). The standard is a regulation in implementing an online learning system. In the national standard, the online learning system is supposed to produce knowledgeable, skillful, and characterized students as the goal of Indonesia's national learning system.

In the current online learning system, social interaction does not occur between students. On the other hand, student character development can be well-formed if there is social interaction between students. Therefore, the government needs to prepare for online learning standards that can develop the character of students.

Besides lacking character development, the current online learning system cannot develop student skills to the fullest. The result of student knowledge needs to be accompanied by the development of student skills. Therefore, the national standard for online education is necessary to emphasize the development of student's abilities. Some studies showed that online learning systems are still difficult to apply for elementary and middle school students. Online Learning should be equipped with face-to-face Learning. Livingstone's research (2012) shows a real advantage for online over face-to-face learning system, even though the effect was more massive for the blended learning system. The blended learning system is a mode of Instruction that combines an online learning system and a face-to-face learning system.

Blended Learning is a learning system suitable for implementation in West Papua during this Pandemic. This Learning is frequently displayed on a continuum, with face-to-face Learning at one extreme and distance learning system at the other extreme (Fresen, 2018). The combination of some aspects of the two extremes generates the blended learning system, located somewhere along the continuum, as presented in Figure 7.

<table>
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<tr>
<th>Face to Face Learning</th>
<th>Blended Learning</th>
<th>Distance Learning</th>
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<td>On campus</td>
<td></td>
<td>Off campus</td>
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Figure 7: Illustration of Blended Learning, adapted from Fresen (2018)

Finally, an online learning system (whole or blended) is successful when carried out by a creative teacher. The teacher is the leading player when learning is implemented using the online learning
system (Goh et al. 2020). ICT alone cannot guarantee positive educational outcomes, but what the technology can achieve in the hands of skilled and imaginative teachers is equality in access to the kinds of teaching and learning resources and constructive interactions (Latchem & Jung, 2010). Furthermore, ICT for human development is not about technology but people using technology (Nawaz & Kundi, 2011). Therefore, the teachers must also be equipped with knowledge and skills to develop e-learning materials creatively and independently. Teachers should be able to act as a center for online learning success.

The efforts to increase teachers' motivation in West Papua to learn and use ICTs for mathematics learning need to be done continuously. It's because teacher motivation plays an essential role in conventional education and e-learning, especially web-based learning (Kao, Wu, & Tsai, 2011; Khanal et al., 2020). Teachers may take some innovative steps in online learning, including developing interactive learning videos and the use of contextual problems in the teaching materials. These creative efforts are expected to improve the productivity of the online learning system.

CONCLUSIONS

There are two main problems in implementing online mathematics learning systems in West Papua, namely accessibility and the ability to use ICT equipment. On the other hand, online mathematics learning is necessary in times of Pandemic COVID-19 and the future. The government and other stakeholders have an essential role in the online mathematics learning system. The government needs to establish a National Standard for Online Education (Distance Education), including improving teachers' abilities and learning tools. However, Blended Learning is a learning system that is suitable to be applied in West Papua during this pandemic.

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What Works and What Does Not: A Reflective Practice on an Online Mathematics Class

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Abstract: As most courses turn into distant formats, what are the benefits and hindrances of conducting university math courses using multiple technologies? In this e-learning environment, are content-neutral platforms necessary for teaching and learning in these classes? Specifically, what activities on the Moodle e-learning platform are more effective than others in supporting undergraduate student math learning and achievement? This paper will report on a reflective practice that examines both qualitative and quantitative datasets in understanding the implemented multimedia distance learning environment at an entry-level math classroom at a large state university in the Midwest and its resulting consequences on the math learning and assessment performance of the students.

Keywords: Postsecondary; Mathematics Education; E-learning; Reflective Practice

INTRODUCTION

When COVID-19 hit, higher education institutes globally experienced an unprecedented shifting from traditional classroom-based instructions to blended or entirely e-learning environments. For Fall 2020, most US colleges and universities adopted the hybrid learning format and offered a mixture of in-person and online learning. While technologies enabled students to continue with their education during this pandemic, researchers had ambivalent attitudes on the topic of e-learning in the subject of mathematics. What technology works, and what does not, when it comes to teaching and learning in a virtual university math course? What activities completed in a multi-technology setting positively impact student assessment performance? Through a practice reflection on my current teaching for two sections of an entry-level university math course at a Midwest public university, this paper employs mixed datasets in exploring the advantages and disadvantages of used content-neutral technology (namely, Moodle, Zoom, and Campuswire) and in testing quantitative connections among activity involvement on the Moodle LCM platform and student performance in three monthly exams.
The paper contains the following sections: Literature Review, which outlines varied views on impacts of technology on education and presents content-specify and content-neutral tools applied in math education; Methodology, which provides a detailed explanation of the methods implemented, the course structure, and participants; Results and Discussion, which contains qualitative and quantitative data obtained from the study and a thorough reflection and reading of the data presented; Conclusion, which summarizes the previous sections and elaborates on future research directions.

LITERATURE REVIEW

Views on Technology in Mathematics Education

As early as the 80s, there was an emerging view of technology as a ground-breaking tool that manifested mathematical investigation and facilitated student development of mathematical thinking and understanding (Shumway, 1989; Fey, 1993). At the seventh International Congresses on Mathematical Education (ICME-7), three workshops and lectures, on Impact of Calculators on Elementary School Curriculum, Technology in Service of the Mathematics Curriculum, and TV in the Mathematics Classroom, are organized to promote technology, especially computers, in modeling and experimenting with mathematical ideas in various school levels. Praises for new technology in the late twentieth century were not limited to calculating X's and Y's. Online modes of communication in mathematics education are also well received by researchers who pointed out that developments in transmissions of graphics, sound, and videos opened the door to an unknown era in distance learning (Knight, 1994). More recently, a group of researchers (Gadanidis & Geiger, 2010; Hughes, 2008) focused their work on the benefits of technology in reconceptualizing social interactions in learning by integrating math performance and collaboration into multi-medias. As Gadanidis and Geiger (2010) mentioned, new technologies and, in particular, various media platforms could bring a common mathematics experience to the general public.

While many researchers were confident towards technology in education, less so were the others. Turkle (2018) argued in her study that technology made us forget what we knew about life. She suggested that technology imperatives were only constructive for bounded populations and that we saw technology as the panacea for many of our education problems ranging from the lack of student engagement to the issue of measuring educational productivity. With the example of online forum participation taken as an always-available discussion, Turkle argued that communications in sizable online courses could have a contrary effect on the learning experience despite what educators intended and believed, based on how, in two MOOCs, interviewees described discussion board and posts as hardly seen or difficult to follow. The viewpoint of e-learning as inferior to face-to-face instructions raised the initial question for this study: If one technology is not accommodating, will having varied technologies add to the diversity and foster a better online classroom experience? This project incorporates both positive and negative perceptions on technology in education while reflecting on whether various technologies are as
prominent as discussed by Gadanidis and Geiger (2010) and whether using more than one type of technology solves the communication problem presented by Turkle (2018).

**Content-specific versus Content-neutral Technologies**

As technology prevails in the field of math teaching, educators adopt all types of aids ranging from computer-based graphic calculators to large-scaled learning management platforms. According to the National Council of Teachers of Mathematics (2011), these technological tools can be categorized as content-specific and content-neutral. The definition of content-specific is revolving around the idea of solving particular mathematical problems, no matter algebraically or geometrically. This category contains computer algebra systems (e.g. Wolfram Alpha), dynamic geometry environments (e.g. Geogebra and Desmos), interactive applets (e.g. Mathematics Vision Project), and data analysis devices (e.g. R and Tableau). Researchers brought forward the notion that, as technology became a part of the learner environment, it was more than substitutions for regular chalrs and boards but a new way of fluid knowledge construction within the field of mathematics (Olive and Makar, 2010). Goos et al. (2010) also proposed that computer algebra systems could support student learning when engaged in mathematical modeling tasks. Understanding and identifying mathematical concepts and relationships is the goal of applying these content-specific instruments.

Contrary to this, content-neutral technologies center on eliciting communications and transmitting the information. Typical content-neutral platforms include Moodle Learning Management System (LMS), Canvas LMS, Blackboard LMS, Compass LMS, Piazza, Campuswire, and Google Classroom. While possessing individual features, all these provide opportunities for collaborative learning and multi-media content delivery in this digital era. This benefit of content-neutral technologies can also positively influence student access to course announcements, materials, interactions, and ultimately ownership of knowledge (NCTM, 2011). The LMS is also identified as a powerful assessment tool. For example, Moodle LMS can contribute to the formative e-assessment of students in math courses by enabling an innovative alternative to the traditional testing system and by providing useful analysis of question difficulty levels through psychometric coefficients of the data stored on the platform (Blanco and Ginoart, 2012). For the purpose of the study, Moodle platform was used for announcements, study reflections, lesson materials, and tests; Zoom meeting tool was used for synchronous sessions; Campuswire forum was used for communications among students and instructors.

Nonetheless, strategic usage of both types of technologies is equally vital as a topic. The teacher and the curriculum play critical roles in technology mediation (Suh, 2010). Without effective lesson plans and carefully designed tool implementations, classes that only consist of unguided appliances are not sufficient in math teaching and learning. Upon the call for practitioner knowledge of technology in math classrooms, this study aims to reflect from the practitioner’s
point of view on conducting an online undergraduate-level math classroom in a content-neutral technology-rich environment.

**METHODOLOGY**

**Research Questions**

The study investigates two aspects of technology in an e-learning undergraduate math course. Firstly, both positive and negative experiences with implemented technology instruments are sought to review my current practice of incorporating Moodle, Zoom, and Campuswire. Secondly, correlations among Moodle activities and assessment results from three monthly exams are explored to analyze the critical relationship between technological tasks and student performance in standardized tests. With an emphasis on these two aspects of the topic, the following two research questions are studied.

1. What are the benefits and hindrances of conducting an online entry-level university course using multiple technologies?

2. How do types of activities (watching asynchronous videos, participating in weekly quizzes, and completing weekly reflection forums) on Moodle correlate to test performance?

**Participants and Procedures**

Since this is teacher research and reflective practice for a single classroom, the reporter is the instructor for the course subject. Twenty-one participants of the course volunteered in this study. The survey, activity, and exam data were collected from these volunteered students. The observations of the classroom were made by perusing the recorded Zoom synchronous meeting sessions. The university that this study took place is a public institute in the Midwest.

This reflective practice project is empirical, involving qualitative and quantitative data collection and interpretation. More specifically, this is a classroom study where the teacher conducts an inquiry into relationships between technology and the classroom with data based on observation, interview, and document collection (Cochran-Smith and Lytle, 1993). This teacher research inquiry is intentional, systematic, public, voluntary, ethical, and contextual (M. Mohr et al., 1994). The project incorporates Gibbs’ cyclical model on the theoretical approach of reflection (Gibbs, 1988). In this six-stage model, the practitioner starts by describing the situation, then expresses their opinions on the matter, followed by evaluating classroom data and analyzing the teaching experience, and ends with a conclusion on possible improvements and an action plan for making advancements. The introduction and literature review sections of this paper correspond to the first two stages; the sections on results and discussion articulate the evaluation and analysis; the conclusion elaborates on the last two stages.
The current data-driven research investigates open questions and does not perceive any preconceptions regarding the effectiveness of technology in math teaching and relationships between technology and math learning. Because no one data source can offer a whole and accurate picture of a complex classroom study, triangulation is applied with multiple perspectives from classroom observations, surveys, Moodle activity logs, and exam data. The mixed types of data obtained in this research are not generalizable because of reflective practice’s characteristics. However, analysis of the data could be transferable to other similar contexts. Upon evaluating the data, the insider perspective secures a more grounded, discovery-oriented, exploratory, expansionist, descriptive, and inductive study (Larsen-Freeman & Long, 1991).

During the 2020 Fall semester, this course on the subject of Finite Mathematics, composed of topics such as basic set theory, probability theory, and matrix theory, is taught at a Midwest public university. Two one-hour synchronous sessions were met through Zoom conference technology on Tuesdays and Thursdays; an extra hour of prerecorded lecture video was posted on Moodle LMS each previous Thursday. In total, ten weekly quizzes and three monthly exams were distributed on Moodle. Each quiz and exam was not cumulative and assessed only on contents from the designated unit. Since the class and all research components were conducted in English only, language should not be an issue in this study.

In addition to classroom observations, a student survey that focused on the multi-technology environment and its impacts on math learning was collected; other collected items included Moodle LMS system data on activity logs and exam scores. The survey asked students five questions that are indicated in Table 1. Moodle system data consisted of assessment scores from three exams (Exam 1, taken in Mid-September, Exam 2, taken in Mid-October, and Exam 3, taken in Mid-November), and activity logs for participation in videos, quizzes, and study reflection forums. All datasets were collected under the consent of voluntarily participated students. The data are analyzed in three steps. First of all, different statistics are applied to quantitative data to test relationships among factors. Next, the results are interpreted and cross-analyzed with qualitative data from the student survey. Lastly, while reflecting on my own teaching experience, I also scrutinize the differences and similarities among different viewpoints from the student survey and connect key points with findings from the previous stages.
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>In our Tuesday lectures, we experimented with both Hybrid learning and E-learning formats. In particular, attendance requirement for synchronous sessions were cancelled starting from Week 9. What are the academic reasons behind your preference for continuing studying in the Hybrid format (synchronous lectures and prerecorded lessons) or your preference for switching into a fully e-learning environment (asynchronous recorded lectures and prerecorded lessons)? Please reply with one or more sentences.</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>In light of cancelling the attendance for synchronous courses, we started to adopt weekly study reflection forums. This forum is meant for you to reflect on your personal study journey regardless of your choice of hybrid or e-learning formats. Were the reflection forums beneficial for your learning in any way? Why or why not? Please reply with one or more sentences.</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>In regular classrooms settings, group study has long been recognized as one of the best way of math learning. In on-line classrooms, however, things could be quite different. In our Thursday sessions, we usually had synchronous group activities. Did you participate in these on-line group activities as much as you did for in-person ones? If you didn’t participate as much, what are some of the reasons? Please reply with one or more sentence.</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>We have used many technologies in this course, ranging from the Moodle platform (for broadcasts, assessments, and forums), the Zoom meeting room (for synchronous and asynchronous sessions), and the Campuswire communication (for student discussion). Please use one sentence or more to reflect on your experience with one or more technology helped you learning in our course.</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>Following from the last question. I have learnt from some students earlier that the current multi-media environment of our course could be overwhelming because of the amount of various tasks they needed to complete using different types of technologies. Do you think having all three technologies mentioned is beneficial to your learning in this course? Or do you think having all of them can present unnecessary burdens in learning? If so, what technologies do you think we should keep using/abandon for this course?</td>
</tr>
</tbody>
</table>

Table 1: Student survey questions
RESULTS AND DISCUSSION

<table>
<thead>
<tr>
<th>Variable</th>
<th>No. of observations</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Min.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Assessment results (out of 100)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exam 1</td>
<td>20</td>
<td>73.625</td>
<td>15.972</td>
<td>44.00</td>
<td>100</td>
</tr>
<tr>
<td>Exam 2</td>
<td>20</td>
<td>92.750</td>
<td>5.720</td>
<td>84.00</td>
<td>100</td>
</tr>
<tr>
<td>Exam 3</td>
<td>21</td>
<td>90.670</td>
<td>11.090</td>
<td>54.06</td>
<td>100</td>
</tr>
<tr>
<td><strong>Moodle Activities</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Videos B. Exam 1</td>
<td>21</td>
<td>5.952</td>
<td>4.177</td>
<td>0.00</td>
<td>12</td>
</tr>
<tr>
<td>Videos B. Exam 2</td>
<td>21</td>
<td>3.667</td>
<td>3.526</td>
<td>0.00</td>
<td>11</td>
</tr>
<tr>
<td>Videos B. Exam 3</td>
<td>21</td>
<td>2.286</td>
<td>2.125</td>
<td>0.00</td>
<td>6</td>
</tr>
<tr>
<td>Forums</td>
<td>21</td>
<td>3.143</td>
<td>1.315</td>
<td>0.00</td>
<td>4</td>
</tr>
<tr>
<td>Quiz</td>
<td>21</td>
<td>9.762</td>
<td>0.768</td>
<td>7.00</td>
<td>10</td>
</tr>
</tbody>
</table>

B: Before.

Table 2: Descriptive statistics for assessment results and Moodle activities
**Groups** | Group 1 (G.1) | Group 2 (G.2) | df  | T-test |
---|---|---|---|---|
[G.1, G.2] | Mean (SD) | Mean (SD) | Welch | T-statistics (p-value) |

**Improvement:** | EX 3 - EX 2 |
---|---|
[S., AS.] | -3.58 (3.54) | -0.21 (7.72) | 12 | t(12) = -1.201 (0.126) |
[AS.V, AS.NV] | -4.97 (11.16) | 4.014 (3.710) | 4 | t(4) = -1.528 (0.100) * |
[F.L, F.] | -6.26 (12.50) | -1.53 (0.81) | 6 | t(6) = -0.926 (0.195) |
[V., V.L] | -7.75 (9.41) | 3.88 (0.79) | 10 | t(10) = -3.893 (0.001) ** |

**Improvement:** | EX 3 - EX 1 |
---|---|
[VT., VT.L] | 12.76 (14.52) | 21.07 (-5.74) | 11 | t(11) = -1.608 (0.068) * |

S.: Attended at least one synchronous session after Exam 2 before Exam 3.
AS.: Attended no synchronous session after Exam 2 before Exam 3.
AS.V: Attended no synchronous session and watched at least one video after Exam 2 before Exam 3.
AS.NV: Attended no synchronous session and watched no video after Exam 2 before Exam 3.
F.L: Posted on not all forums after Exam 2 before Exam 3.
F.: Posted on all forums after Exam 2 before Exam 3.
V.: Watched more than one videos after Exam 2 before Exam 3.
V.L: Watched no or only one video after Exam 2 before Exam 3.
VT.: Watched more than or equal to ten videos in total before Exam 3.
VT.L: Watched less than ten videos in total before Exam 3.

*Note* *p < 0.11; **p < 0.0011

Table 3: T-tests for pairs of groups based on their technology usage and academic improvement
<table>
<thead>
<tr>
<th>Items</th>
<th>Yes</th>
<th>NA</th>
<th>No</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Technology</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moodle</td>
<td>15</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Campuswire</td>
<td>1</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Zoom</td>
<td>9</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td><strong>Activity</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Forum</td>
<td>8</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>Async. Format</td>
<td>14</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Group Activity</td>
<td>3</td>
<td>0</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 4: Student survey data

Figure 1: Student survey chart
RELATIONSHIP BETWEEN MIXED VS. ASYNCHRONOUS E-LEARNING AND IMPROVEMENT

Firstly, students are assigned into two groups based on whether they chose to continue studying synchronously or switch to learning asynchronously: Group S. with at least one synchronous session attended; Group AS. with none synchronous session attended. Then, a T-test is conducted to compare the mean improvement in exam scores between these two groups of students. The null hypothesis is that there is no difference in their improvement, and the alternative hypothesis is that the students who attended asynchronously have higher improvement scores. The results are displayed in Table 4.

One-sided two-sample mean Welch’s T-tests are employed. The mean score differences from Exam 3 to Exam 2 (t(12) = -1.201, p < 0.15) are relatively more negative for Group S. students who are taking synchronous sessions. Thus, there is some statistical evidence rejecting the null hypothesis. The statistics conclude that students who participate synchronously after Exam 2 before Exam 3 do not necessarily tend to have more developments, compared to their peers who participate asynchronously.

Therefore, it is arguable that pure non-concurrent e-learning is more beneficial to assessment performance than in a mixed manner where students need to manage both synchronous and asynchronous videos. This result could be related to the concept of Zoom fatigue, where students experience tiredness from attending concurrent sessions due to not being able to determine facial expressions and caring too much about self-appearance on camera along with other internet issues. Even though this comparative profit of fully asynchronous learning is displayed, there is value in having both synchronous and asynchronous options available for students. This is also mentioned by Student X, shown in Excerpt 1.

“I like how both options are available. At points I fell behind and it would not have been beneficial for me to go to the live lecture because I hadn’t seen the previous lecture, watched the instructional videos, or maybe I had gone to bed really late and I needed the sleep. However, when I am caught up, it is better for me to watch the live lecture because it is much more time-efficient since I can’t pause it and get distracted by my phone or something else.”

Excerpt 1: Student X’s opinion on having both synchronous and asynchronous options available

To further explore the reason behind this, four similar T-tests are performed on the following pairs: 1. Attended no synchronous session and watched at least one video after Exam 2 before Exam 3 (AS.V) vs. Attended no synchronous session and watched no video after Exam 2 before Exam 3 (AS.NV); 2. Posted reflections on not all forums after Exam 2 before Exam 3 (F.L) vs. Posted reflections on all forums after Exam 2 before Exam 3 (F.); 3. Watched more than one videos after Exam 2 before Exam 3 (V.) vs. Watched no or only one video after Exam 2 before Exam 3 (V.L); 4. Watched more than or equal to ten videos in total before Exam 3 (VT.) vs.
Watched less than ten videos in total before Exam 3 (VT.L). The results from these four findings are discussed in more detail in the sections below.

**RELATIONSHIP BETWEEN MORE VIDEOS VS. FEWER VIDEOS WATCHED AND IMPROVEMENT**

From test statistics in Table 3, there is relatively significant statistical evidence at 0.1 rejecting the null hypothesis for AS.V and AS.NV. The statistics conclude that students who do not participate in Zoom concurrent meetings nor recorded videos after Exam 2 before Exam 3 tend to have more improvements compared to their peers who do watch recorded videos while taking the course in an asynchronous format. Within the group of asynchronous-only learners, it is suggested that not watching as many videos correlates to the increase in student improvements. This could result from the confusing layout of videos. Each week, instructors posted two sets of pre-lecture videos on the same contents and a recorded Zoom session. Originally, the repetitions and similar explanations of concepts and problems were designed to provide students with a non-uniform learning experience, which, in turn, should promote student growth academically. However, the result says otherwise.

There is also highly significant statistical evidence at 0.001 rejecting the null hypothesis for V. versus V.L. The statistics conclude that students who watch fewer than two recorded videos after Exam 2 before Exam 3 tend to have more improvements compared to their peers who do watch more than one recorded video during the same period. This fact reveals that having fewer recorded videos required-to-be-watched is better in assisting student learning inquiry, which is in parallel with the viewpoint demonstrated in AS. V vs. AS.NV. In a similar vein, there is some statistical evidence at 0.1 disapproving of the null hypothesis, when it comes to the pair VT. and VT.L. The statistics conclude that students who watch fewer than eleven videos in total before Exam 3 tend to grow more academically in math, compared to their peers who watch more than ten recorded videos throughout the three months. This information extends the former findings.

On top of this, upon analyzing relevant qualitative data from the survey, Videos is not a factor positively influencing improvement in both mixed and asynchronous e-learning settings. For the mentioned five students who switched to asynchronous study without videos, their growth in academic achievement is relevant to their satisfying experience with the content-neutral aspect of Moodle LMC. All five students mentioned lucid structures of the course Moodle page and discussed how they prefer fewer technologies and focus on only Moodle non-videos resource as a course tool.

In Group S., there is one student who has the same improvement score as the previous five Group AS. students. This student also mentioned enjoying Moodle non-video resources as the only tool for the course. These two cases demonstrate that students can benefit from not viewing as many videos on Moodle because they chose to concentrate on fewer tasks at a time, which could lead to higher study efficiency and learning outcomes. Fewer recorded videos should be uploaded
to Moodle to enforce a more minimal layout. However, it is noteworthy that the findings are not significant enough to reject all the synchronous and asynchronous sessions. The course should still provide students with learning opportunities via media, but just not as many to the state that can burdensome for students to manage.

RELATIONSHIP BETWEEN FEWER FORUMS VS. ALL FORUMS COMPLETION AND IMPROVEMENT

For the groups, F.L and F., there exist some minor significance at 0.2 rejecting the null hypothesis, as illustrated in Table 3. The statistics conclude that students who do post all four weekly study reflection forums after Exam 2 before Exam 3 tend to perform more ideally, compared to their peers who are not involved in as many posts. This finding shows that, by completing all weekly study reflections, it is likely for students to improve more in exam scores.

Contrary to our finding that forum is a factor in increasing exam scores, from Table 4, more than half of the students thought of the forum as not helpful in academic improvement. From the twelve students’ reasons for not perceiving Forum as a useful activity, many pointed out that it only asks for what they have done but does not provoke any deep thoughts and considerations. For reference, the question in Excerpt 2 is used in all forums. Based on the positive effects Forum has on student math learning, this activity can still be maintained but surely with changes in question wordings. In light of this, it may also be advantageous to have multiple versions of questions to help facilitate a less static and more active learning environment.

“Please reflect on your learning this week and write a sentence or two describing one or more of the following: What have you studied? What appears to be interesting/difficult? Are there any questions regarding this week’s materials?”

Excerpt 2: Questions on four weekly reflection forums

TEACHING REFLECTION

Multiple technologies for this online mathematics course benefited the class by presenting diverse activities and enabling the switch from synchronous mandatory to synchronous optional. These content-neutral tools also allowed me to engage in more conversations and closer connections with my students regardless of the distance between us. Meanwhile, reducing the technologies, in the second half of the semester for this course, made positive impacts on student learning. By choosing either Zoom and Moodle (synchronous lectures) or only Moodle (asynchronous recordings), students could invest energy in their ideal way of learning. With fewer technologies required for the course, they also had a clearer vision of how to master the materials. Before the schedule altered, the course contained multiple, even an excessive amount of, obligatory activities and systems that induced much confusion. After the change, students had less workload and also took part in course structure decisions.
Another tool Campuswire Discussion Platform (DP), with powerful messaging and grouping tools, was planned to engender conversations among students and ultimately build a mathematical community for the course. Sadly, barely any student was willing to participate in study groups and extra activities for this course, which might be because of the already heavy workload in- and out-side of the class. Based on survey data from Table 4, half of the students voted for abandoning this technology. Several of them suggested using the conversation and group feature on Moodle instead. Through this three-month teaching, I had not been a frequent user of Campuswire either. The inconsistency and disconnection, between this DP and the most used platform Moodle, is the problem. It is indeed time- and energy-consuming to monitor all the conversations that happened sparsely over the space, which is similar to the previous scenario with multiple technologies for one course.

As the only highly praised technology by over half of the students in the questionnaire, shown in Table 4, Moodle LMC played a vital role in my teaching. It provided a common platform for everyone to share course resources, including activity problems, videos, Zoom links, and forums. The fact that all materials could be easily found on the course website lessened tasks for students and provided a more organized environment for concentrating on the course materials. Among the Moodle activities, Videos was found in previous sections to be less effective in elicit testing improvement. This is associated with the large volume of videos available for students and the mentioned repetitions of video contents. While one student indicated having two sets of videos was beneficial, others pointed out the redundancy within these videos which resulted in their uncertainty of what was more important. In alignment with the previous discussion, either the number of videos should be reduced, or some clear identifications on the importance of videos should be given. With fewer items to concentrate on, students can improve more efficiently in assessment performance.

CONCLUSION

Reflecting on my teaching experience, I have gained deeper understandings of the benefits and hindrances of having multiple technologies for math online learning. With mixed types of data, the study shows that students prefer to use fewer technologies for this online course due to the deducted difficulty in management. This is in alignment with the finding that the higher the number of content-neutral tools is, the lower student engagement is. Among the multiple tools used for this class, Moodle LMS was the most used because of its clarity and functionality. A majority of course contents were delivered on Moodle with its activity features, especially in the second half of the semester, where asynchronous videos were uploaded as Moodle videos. Both the instructor and the students identified this LMS platform as easily accessible. Zoom meeting tool was mostly used for holding synchronous sessions and recording the lecture videos. It is beginner-friendly but has only two functions: meeting and chat. For the purpose of this study, the chat function of Zoom was not investigated. Zoom, by itself, cannot deliver this undergraduate math course completely.
online, since vital features (announcements, permanent file uploads, etc.) are missing. Campuswire, different from the previous two, was not used often in this course. The main reason is the overlapping of its conversation feature with the one on Moodle. With the forum and chat option available on Moodle, students did not see the need to learn new technology and spend effort managing it.

Besides analysis on the multi-technology setting for the course, I also discovered correlations between Moodle activity involvement and student examination performance. From the quantitative data collected, an appropriate number of pre-lecture and lecture videos are in need to assist students with academic improvement. It is surprising that the higher the number of videos watched, the lower the degree of improvement is seen in standardized testing. Considering this reflection, the content and organization of Videos should be adjusted to accommodate only key theorems and questions without overlapping materials. Unlike the strong correlation displayed, a minor relationship exists between forum participation and academic achievement. This low level of correlation could be from the lack of data. The forum activity was only implemented for the last four weeks of the term. In the future, the course will start the study reflection forum activity from Week 1. In this way, more data can be analyzed to determine a more explicit relationship. On the other hand, weekly quiz participation does not provide interesting results for this study, owing to the fact that all quizzes were completed by everyone and cannot be viewed as a variable for statistical tests.

In conclusion, less is more. This study on an online undergraduate math course reveals that the multi-technology environment is not as fruitful as many theoretical researchers described. Even though there exist limitations due to the nature of the reflective practice, it is evident that, in this course, students improve much more in test performance when using fewer technologies and tools. For my future teaching, I plan to keep Moodle LMC, remove Campuswire, and make changes to activities like Videos and Forum. These updates will ensure a more straightforward and manageable virtual learning environment for students, while creating more quality opportunities for student performance improvement in standardized tests. Further studies on this topic can explore the associations between participation in activities on Moodle and performance in weekly quizzes.

References


Teaching and learning of mathematics and criteria for its improvement from the perspective of future teachers: a view from the Ontosemiotic Approach

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Abstract: The objective of this article is to identify the meaning attributed to the didactics of mathematics and what are the criteria with which an improvement in the teaching and learning process of mathematics is based, future teachers of mathematics, belonging to universities in three different countries (Brazil, Chile, and Ecuador). The qualitative analysis indicates that the majority of future teachers consider that the didactics of mathematics is a technical discipline that consists of providing strategies, resources, and procedures for teaching mathematics; few consider it as an art to teaching and almost none consider it as a scientific discipline that is concerned with studying the processes of teaching and learning mathematics. In addition, the results show that the criteria used by them, on how teaching and learning in this discipline can be improved, are focus, above all, in the cognitive, ecological, and emotional aspects and, to a lesser extent, to the interactional, mediational and epistemic. Finally, it is concluded that the improvement in teaching and learning is directly related to an improvement in the training programs of future mathematics teachers.

INTRODUCTION

There are different characteristics that the training and professional development programs of mathematics teachers must possess. One of the topics of discussion in mathematics didactics is directly or indirectly related to improving teaching and learning processes. A goal of these teacher training programs is to achieve an impact on the improvement of mathematics teaching. This aspect generates the need to understand the following questions: What is understood by the process of improving the teaching and learning of mathematics? And what is the role of mathematics didactics in supporting this process?
There are different improvements that can be considered in the area of mathematics didactics. The first can be perceived as training programs. A second way is to follow certain trends in the teaching of mathematics, considering that this can generate improvements (Bishop, Clements, Keitel, Kilpatrick, and Leung, 2003; English, Bartolini-Busi, Jones, Lesh, and Tirosh, 2008; Gutiérrez and Boero, 2006; Lester, 2007). On the other hand, a third way is to follow some principles and patterns that guide the development of the practice of mathematics teaching (NCTM, 2000; Breda, Font, and Pino-Fan, 2018). These principles were and are generated by the community interested in mathematics education. In particular, the community of researchers in the Didactics of Mathematics and its scientific contributions, and also the educational community in general (teachers, students, parents, administration, etc.).

Some studies have addressed the role of Mathematics Didactics (MD), focusing on understanding the conceptions about MD in trainers of future teachers and the development of didactic-mathematical skills or abilities in future teachers (Oliveira and Fiorentini, 2018; Nortes and Nortes, 2011; Rosa, Farsani, and Silva, 2020), others have developed research focused on the beliefs and conceptions of future teachers about mathematics education and the teaching and learning processes of that discipline (Gvozdic and Sander, 2018; Marbán, Palacios and Maroto, 2020; Manderfeld and Siller, 2019) and others have focused their interest on the meaning given to MD from the teaching experiences experienced by classroom teachers (Zumaeta, Fuster and Ocaña, 2018; Farsani, 2015; 2016). Also, there are researchers who have been concerned with studying how MD is constituted as a research field. Some of the conclusions they have reached is that the Didactics of Mathematics is constituted in the middle of power strategies. It operates in different ways and is not subordinate to other disciplines (Fernandes, 2014).

In line with the aforementioned, this article aims to identify the meaning that future mathematics teachers from three Latin American countries (Brazil, Chile, and Ecuador) attribute to the didactics of mathematics and the criteria with which they base an improvement in the process of teaching and learning mathematics.

In the following sections we present: a) The theoretical framework, which explains the different understandings about what MD is and its role as a discipline, the idea of improvement in the teaching and learning of mathematics and, finally, the construct Didactical Suitability Criteria (DSC) of the Ontosemiotic Approach (OSA); b) The methodology, which presents the context of the study, the instrument with which the data were collected and the procedure for analysing them; c) The results and discussions; d) The considerations of this investigation.

**LITERATURE REVIEW**

In this section, we will present a brief description of what MD is and its research from the perspective of different theorists in the area, as well as what we consider to be an
improvement in the teaching and learning processes of mathematics. We finish by explaining the construct that we have chosen as a tool to prescribe, evaluate and analyze the teaching and learning processes of the discipline under study, the DSC.

**Didactics of Mathematics**

Since its inception (Steiner, 1985; Brousseau, 1989), research on the teaching of mathematics was shaped by the field of educational research, which consequently changed its initial focus of philosophical speculations. The didactics of mathematics is an art of teaching, which is a process used by teachers helping their students develop mathematical skills and new knowledge (Kilpatrick, 1998). In particular, the evolution of MD, as a result of its direct relationship with changes in research on mathematical knowledge, has led it to try to characterize itself as a scientific discipline (Gascón, 1998; Gascón and Nicolás, 2017). Steiner (1985, p.11) observed that:

Mathematics Education can never become a science or a field with scientific foundations. The field is then left open for highly subjective views and beliefs, for short range pragmatism and an interpretation of mathematics teaching as primarily an art.

The scientific character, as Brousseau (1989) points out, is classified into a) the applied multidisciplinary conception - serves to instruct the necessary teachings for the professional training of teachers and as a field of research carried out on teaching considering the scientific disciplines such as psychology, pedagogy, sociology, semiotics, etc. b) the autonomous conception - fundamental of the discipline itself, mathematics.

The concepts presented lead us to understand MD as a scientific or technical discipline, endowed with methodological aspects. These methodological aspects serve to explain how the teaching and learning processes of mathematics are carried out without referring to prescriptive aspects or evaluative.

On the other hand, Schoenfeld (2000) at the beginning of the 21st century, when proposing his questions concerning the nature of research in the didactics of mathematics, argues that it has two main purposes, one pure and the other applied. For this author, the pure is related, above all, in understanding the nature of mathematical thinking, teaching, and learning, while the applied purpose is related, above all, in using that understanding to improve mathematical instruction. Godino (2006; 2010) corroborates the idea of Schoenfeld (2000) and the idea that MD is a scientific discipline made up of three large fields: a) Didactic technology, b) Scientific research and, c) Practical and reflective action. Didactic technology aims to develop materials and resources, using available scientific knowledge. Scientific research tries to understand the functioning of the teaching of mathematics as a whole. Finally, practical and reflective action examines the teaching and learning processes of mathematics. For this author, these three fields are focused on the operation of didactic systems and have an ultimate purpose: the improvement of the teaching and learning of mathematics. With the same purpose, Lesh and Sriramn (2010),
consider MD as science-oriented to the design of processes and resources to improve the teaching and learning processes of mathematics.

Godino, Batanero, and Font (2019) consider that MD has a scientific and technological character, thus demonstrating a broad conception of MD as a scientific discipline, since it must consider that it must address theoretical issues of mathematical knowledge itself (its ontological characteristics, epistemological, semiotic), descriptive, explanatory, predictive questions (relationships of the theoretical questions of mathematical knowledge with the teaching and learning processes), typical of scientific knowledge, and also prescriptive and evaluative questions, typical of technological knowledge.

The ideas presented lead us to classify the different understandings related to MD. The first is to understand MD as an art of teaching, which takes away the scientific character of the discipline. The second is to understand MD as a scientific discipline that is based on methods and theories that help us describe and explain how mathematical knowledge is generated and how its teaching and learning processes are developed. Finally, the third is to understand MD as science-oriented towards the improvement of the teaching and learning processes of mathematics, that is, its prescriptive and evaluative characteristics.

**Improvement in the teaching of mathematics: philosophical aspects**

Considering the MD perspective that leans towards prescriptive and/or evaluative aspects, in this section, what is understood by an improvement of mathematics teaching and learning processes is clarified.

The idea of improving teaching and learning processes is related, above all, to the idea of truth. That is, the truth about what is considered as process improvement can come, from philosophy, starting from a positivist or consensual perspective (Nicolás, 1997).

From the positivist perspective, what is correct, incorrect, good, bad, has quality (or not) will be told by the progress of this scientific area, which will find objective results that will guide us to improve the teaching and learning processes. At its core, it is a positivist discourse based on the theory of truth as correspondence (Font and Godino, 2011). From this point of view, the strategy to improve the teaching and learning processes of mathematics should be of the top/bottom type. Now, the main problem with this way of understanding change and improvement is that teachers are not included in the process, they are limited to applying curricular materials designed by experts dedicated to research. This perspective, while giving great importance to the role of theory, limits the role of the teacher to that of the user and does not take much into account the socio-political factors that affect mathematics education.

The consensual perspective is a notion inspired by the idea of Peirce’s consensual theory of truth and its developments by Apel and Habermas (Nicolás, 1997). From this perspective, the Didactics of Mathematics can offer us provisional principles agreed by the interested community, which can...
serve to guide and assess the teaching and learning processes of mathematics. As explained in Breda, Font, and Pino-Fan (2018), trends in mathematics teaching are a first way of observing consensus in the Educational Mathematics community, since they can be considered as regularities found in the speeches on the improvement of the teaching of mathematics (Guzmán, 2007). A second way of observing consensus is the reconversion of some of these trends into explicit principles, such as the principles and standards of the National Council of Teachers of Mathematics (NCTM, 2000), which emerge from a broad consensus among teachers, the association of teachers, trainers of mathematics teachers, representatives of the educational administrations, researchers and mathematicians, all of them with extensive educational experience.

In addition to that, in the Didactics of Mathematics knowledge and results have been generated that enjoy wide consensus. Along these lines, a characteristic of many theoretical approaches to the area is that, in addition to assuming some principles for the development of their theoretical construction, they consider that these principles should be considered in the teaching of mathematics so that it is better.

For the development of the didactical suitability construct, current trends on the teaching of mathematics, the NCTM principles, and the contributions of the different theoretical approaches in the area of Mathematics Didactics have been considered (Godino, 2013; Breda, Font, and Pino-Fan, 2018).

In the OSA, the didactical suitability criteria (DSC), its components, and characteristics were constructed on the basis that they should be constructs that rely on a certain amount of consensus within the Mathematics Education community. As a result, it was considered that, given the ample consensus they generate, the principles of the NCTM (2000) could serve as the basis for some of the DSC, or rather, they could be considered as some of the components themselves. On the other hand, for the development of the didactical suitability construct, some of the contributions (principles, results, etc.) of the different approaches of the Mathematics Education area were also taken into account (Godino, 2013). Therefore, one of the plausible explanations that the suitability criteria can be considered as teachers’ reflection patterns is related to the extensive consensus that they generate amongst persons involved in Mathematics Education (Breda, Pino-Fan and Font, 2017).

Didactical Suitability Criteria

The DSC arises in response to the following question: What kind of actions and resources should be implemented in the instructional processes to optimize mathematical learning? According to the OSA, the notion of didactical suitability is defined as the degree to which the process (or a part of it) meets certain characteristics that allow it to be classified as optimal or adequate to achieve adaptation between the personal meanings achieved by the students (learning) and the institutional meanings intended or implemented (teaching), considering the circumstances and available
resources (environment) (Godino, Batanero and Font, 2019). As we pointed out previously, DSC can serve, a priori, to guide (or plan) the teaching and learning processes of mathematics and, in the aftermath, to assess their implementations. The OSA considers the following DSC (Font, Planas, and Godino, 2010):

1. Epistemic suitability, to assess whether the mathematics being taught is “good mathematics”.

2. Cognitive suitability, to assess, before starting the instructional process, if what is to be taught is at a reasonable distance from what the students know, and after the process, if the acquired learning is close to what was intended to teach.

3. Interactional suitability, to assess whether the interactions resolve doubts and difficulties of the students.

4. Mediational suitability, to assess the adequacy of the material and temporal resources used in the instructional process.

5. Emotional Suitability, to assess the involvement (interests and motivations) of students during the instructional process.

6. Ecological Suitability, to assess the adequacy of the instructional process to the educational project of the center, to the curricular guidelines, and the conditions of the social and professional environment.

The operation of the DSC requires defining a set of observable indicators, which allow assessing the degree of suitability of each of these criteria. For example, there is consensus that it is necessary to implement "good" mathematics, but it is possible to understand very different things for it. For some DSC, the indicators are relatively easy to agree on (for example, for the criterion of the suitability of means), for others, as is the case of epistemic suitability, it is more difficult. Once the six criteria of partial suitability have been determined, each one of them is broken down into components and indicators, which make them operational. Tables 1, 2, 3, 4, 5, and 6 present the DSC, published in Breda, Pino-Fan, and Font (2017).
<table>
<thead>
<tr>
<th>Components</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Errors</td>
<td>✓ Practices considered mathematically incorrect are not observed.</td>
</tr>
<tr>
<td>Ambiguities</td>
<td>✓ Ambiguities that could confuse students are not observed; definitions and procedures are clear and correctly expressed, and adapted to the target level of education; explanations, evidence and demonstrations are suitable for the target level of education, the use of metaphors is controlled, etc.</td>
</tr>
<tr>
<td>Diversity of processes</td>
<td>✓ Relevant processes in mathematical activity (modelling, argumentation, problem-solving, connections, etc.) are considered in the sequence of tasks.</td>
</tr>
<tr>
<td>Representation</td>
<td>✓ The partial meanings (constituted of definitions, properties, procedures, etc.), are representative samples of the complexity of the mathematical notion chosen to be taught as part of the curriculum.</td>
</tr>
<tr>
<td></td>
<td>✓ For one or more partial meanings, a representative sample of problems is provided.</td>
</tr>
<tr>
<td></td>
<td>✓ The use of different modes of expression (verbal, graphic, symbolic...), treatments and conversations amongst students are part of one or more of the constituents of partial sense.</td>
</tr>
</tbody>
</table>

Table 1: Components and characteristics of epistemic suitability.

<table>
<thead>
<tr>
<th>Components</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Previous knowledge (similar components to those of epistemic suitability)</td>
<td>✓ Students have the necessary previous knowledge to study the topic (that is, they have previously studied or the teacher makes a study plan).</td>
</tr>
<tr>
<td></td>
<td>✓ The intended meanings (reasonable difficulty) can be taught through its diverse components.</td>
</tr>
<tr>
<td>Adaptation of the curriculum to the individuals’ different needs</td>
<td>✓ Development and support activities are included.</td>
</tr>
</tbody>
</table>
Learning ✓ The diverse methods of evaluation demonstrate the application of intended or implemented knowledge/competences.

High cognitive demand ✓ Relevant cognitive processes are activated (generalization, intra-mathematical connections, changes of representations, speculations, etc.)

✓ Metacognitive processes are promoted.

Table 2: Components and characteristics of cognitive suitability.

<table>
<thead>
<tr>
<th>Components</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher-student interaction</td>
<td>✓ The teacher appropriately presents the topic (clear and well-organized presentation, not speaking too fast, emphasis on the key concept of the topic, etc.)</td>
</tr>
<tr>
<td></td>
<td>✓ Students’ conflicts of sense are recognized and resolved (students’ silence, facial expressions, questions are correctly interpreted and an appropriate survey is conducted, etc.)</td>
</tr>
<tr>
<td></td>
<td>✓ The aim is to reach a consensus on the best argument.</td>
</tr>
<tr>
<td></td>
<td>✓ Varieties of rhetorical and rational devices are used to involve the students and capture their attention.</td>
</tr>
<tr>
<td></td>
<td>✓ The inclusion of students into the class dynamic is facilitated – exclusion is not.</td>
</tr>
</tbody>
</table>

Interaction amongst learners ✓ Dialogue and communication between students is encouraged. Inclusion in the group is preferred and exclusion is discouraged.

Autonomy ✓ Moments in which students take on responsibility for their study (exploration, formulation and validation) are observed.

Formative evaluation ✓ Systematic observation of the cognitive progress of the students.

Table 3: Components and characteristics of interactional suitability.
Components and characteristics of mediational suitability.

<table>
<thead>
<tr>
<th>Components</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material resources (manipulatives, calculators, computers)</td>
<td>✓ The use of manipulatives and technology, which give way to favorable conditions, language, procedures, and arguments, adapted to the intended sense.</td>
</tr>
<tr>
<td></td>
<td>✓ Definitions and properties are contextualized and motivated using concrete situations, models, and visualizations.</td>
</tr>
<tr>
<td>Number of students, scheduling, classroom conditions</td>
<td>✓ The number and distribution of students enables the desired teaching to take place.</td>
</tr>
<tr>
<td></td>
<td>✓ The timetable of the course is appropriate (for example, not all the classes are held late)</td>
</tr>
<tr>
<td></td>
<td>✓ The classroom and the distribution of the students is appropriate for the development of the intended instructional method.</td>
</tr>
<tr>
<td>Time (for group teaching/tutorials; time for learning)</td>
<td>✓ Accommodating the intended/implemented content to the available time (contact or non-contact hours)</td>
</tr>
<tr>
<td></td>
<td>✓ Devotion of time to the most important or central aspects of the topic.</td>
</tr>
<tr>
<td></td>
<td>✓ Devotion of time to topic areas that present more difficulty.</td>
</tr>
</tbody>
</table>

Table 4: Components and characteristics of mediational suitability.

Components and characteristics of affective suitability.

<table>
<thead>
<tr>
<th>Components</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interests and needs</td>
<td>The selection of tasks that are of interest to the students.</td>
</tr>
<tr>
<td></td>
<td>Introduction of scenarios that enable students to evaluate the practicality of mathematics in everyday situations and professional life.</td>
</tr>
<tr>
<td>Attitudes</td>
<td>Promoting involvement in activities, perseverance, responsibility, etc.</td>
</tr>
<tr>
<td></td>
<td>Reasoning should be done so in a context of equality; the argument will be valued in its own right and not by the person who puts it forward.</td>
</tr>
<tr>
<td>Emotions</td>
<td>Promotion of self-esteem, avoiding rejection, phobia or fear of mathematics.</td>
</tr>
<tr>
<td></td>
<td>Aesthetic qualities and the precision of mathematics are emphasized.</td>
</tr>
</tbody>
</table>

Table 5: Components and characteristics of affective suitability.
Adaptation to the curriculum ✓ The content, its implementation and evaluation, correspond to the curricular plan.

Intra/Interdisciplinary connections ✓ The content is related to other mathematical topics (connection of advanced mathematics with curricular mathematics and the connection between different mathematics content covered in the curriculum) or to the content of other disciplines, (an extra-mathematical context or rather links with other subjects from the same educational stage).

Social-professional practicality ✓ The course content is useful for socio-professional insertion.

Didactical Innovation ✓ Innovation based on reflexive research and practice (introduction of new content, technological resources, methods of evaluation, classroom organization, etc.)

Table 6: Components and characteristics of ecological suitability.

As mentioned, both the components and the indicators of the DSC have been made considering the consensual perspective of the truth, that is, the trends, principles, and results of research in the area of Didactics of Mathematics. Particularly, for epistemic suitability, this principle has been considered. Mathematical objects emerge from practices, which embodies their complexity (Font, Godino and Gallardo, 2013; Rondero and Font, 2015). From this principle, a component is derived (representativeness of complexity) whose objective is to consider the mathematical complexity in the design and redesign of the didactic sequences (Pino-Fan, Castro, Godino, and Font, 2013; Monje, Seckel, and Breda, 2018).

METHOD

To identify the meaning they attribute to mathematics didactics and the criteria with which they base an improvement in the mathematics teaching process, a qualitative research methodology has been used (Lüdke and André, 1986). In what follows, we explain the research context, the data collection instrument, and the data analysis process.

Research context

This study was conducted with forty-nine students (future teachers) who were studying Basic General Education with a major and a Bachelor's degree in Mathematics in three Latin American countries: Brazil, Chile, and Ecuador. The students of the Basic General Education program with a mention belong to a Chilean subsidized university and an Ecuadorian public university. For their part, the students of the Bachelor's degree in Mathematics belong to a Brazilian public university.
The twenty-four future professors of the Ecuadorian public university, located in the region of Azogues in southern Ecuador, were studying the fifth semester (approximately in the middle of the course) of the Basic General Pedagogy career - which presents a total of nine semesters and medium. In this career, the choice to mention in a specific discipline (mathematics or language) is available from the seventh semester. In this sense, the participants were taking, for the first time, an introductory mathematics didactic course called Teaching and Learning Mathematics I, whose objective was to know, analyze and design strategies and resources in the area of mathematics. It is important to emphasize that, until the fifth semester, the students had not taken any specific subject in mathematics or mathematics didactics, but they had taken pedagogy and pre-professional practices from the first semester of the program or career.

The seventeen future professors from the Brazilian public university, located in the Minas Gerais region in south-eastern Brazil, had already completed nearly 50% of the Bachelor's degree in Mathematics - which has a total of eight semesters. The students had already taken the disciplines of Pedagogical Practices in the Teaching of Mathematics, General Didactics, Teaching of ‘Greatness and Measurement, Teaching of Geometry’, Teaching of Statistics and Financial Mathematics, Computational Resources, Laboratory of Teaching of Mathematics I, and Supervised Practice I. Also, they had taken specific subjects in mathematics (geometry, algebra, and calculus). Of the seventeen participants in this research, fifteen participated in an introductory teaching project promoted by the Teaching Initiation Scholarship Program (or PIBID in Programa de Bolsas de Iniciação à Docência).

The group of eight future professors from the Chilean subsidized university (both public and private), located in the Maule region in central-south Chile, were studying the seventh semester of the study plan of the Basic General Pedagogy career with a mention in Mathematics - which presents a total of ten semesters. Until the seventh semester, the study participants had already taken five mathematics subjects, each with a didactic approach. It should be noted that the training plan for these participants includes internships in schools from the fourth semester.

**Data collection**

The data were collected through a questionnaire of four following open questions: a) What do you understand by Didactics of Mathematics? b) What kind of questions should the Didactics of Mathematics answer? c) What does an improvement in the teaching of mathematics mean? and d) How to develop didactic proposals that represent an improvement in the teaching and learning of mathematics? In particular, the first two questions of the instrument tried to find information about what future teachers understood by Didactics of Mathematics and the last two tried to collect information about the meaning of improvement in the teaching and learning of mathematics attributed by them.
Data analysis

To analyze the participants' discourse and identify what they understand by Mathematics Didactics, we have considered the a priori categories presented in the section on Mathematics Didactics present in the theoretical framework. That is, the responses of the future teachers were categorized considering the different understandings of MD present in the literature: the art of teaching, scientific discipline and/or technological discipline.

On the other hand, to analyze the discourse of the participants when they refer to what is an improvement in the teaching and learning of mathematics and how one can develop didactic proposals that represent an improvement concerning the teaching of mathematics that is usually carried out, a priori categories have also been used. The categories that we have considered are the didactical suitability criteria proposed by the OSA (epistemic, mediational, ecological, emotional, interactional, and cognitive suitability) and its components and indicators. The reason for the use of taking DSC as categories of analysis comes from the constitution of the construct itself (Breda, Font and Pino-Fan, 2018), in particular, it is based on the implicit or explicit assumption that there are certain trends related to the mathematics teaching that indicate how an improvement in mathematics teaching should be compared to what is usually done (Font, 2008; Guzmán, 2007). In particular, it was sought to categorize according to the DSC, which are the criteria assumed by future teachers when they declare how the improvement of mathematics teaching should be carried out.

RESULTS AND DISCUSSION

When analyzing the responses of the forty-nine future teachers about what meaning they attribute to MD, we have observed that twenty-eight of them understand MD as a technological discipline, that is, as a set of techniques, procedures, and resources that serve to improve the teaching and learning of mathematics. For example, as evidenced in the following units of analysis:

*The didactics of mathematics are the methods in which teachers develop the attitudes, skills, and knowledge of students for their teaching and learning (Student 4, Chile).*

*It is a set of actions, applied by teachers in the classroom, which serves to instruct students in mathematical knowledge (Student 17, Brazil).*

*I understand that mathematics didactics refers to how and with which the teaching and learning of mathematics can be improved and students can understand the contents more easily. I also understand that these are strategies that can be applied within the classroom that respond to the students' needs (Student 1, Ecuador).*

Three understand it as an art of teaching, not considering the scientific or technological aspect of the discipline:
Didactics is related to how mathematics can be taught, that is, how we students teach subjects related to mathematics. Didactics is an art and in mathematics, it is an art of reaching students understandably. (Student 5, Ecuador).

For their part, three future teachers understand it as a scientific discipline that is concerned, in addition to the techniques, procedures, and resources, of studying the teaching and learning processes of mathematics:

MD is a set of teaching and learning in conjunction with pedagogy and psychology. It is understood as the relationship between theory and practice in which it leads the student to think, create and build (Student 10, Brazil).

The didactics of mathematics refers to the study of the teaching and learning processes of mathematical sciences. (Student 6, Ecuador).

The results obtained in this first topic allow us to observe that the responses are very similar in the three countries that have participated in the research. In other words, of the seventeen participants from the Brazilian university, fourteen assume that MD is a technical discipline, two consider it the art of teaching and one considers it a scientific discipline. On the other hand, of the twenty-four participants from the University of Ecuador, nine have answered this question, where it is observed that seven consider MD as a technical discipline, one considers it the art of teaching and one considers it a scientific discipline. Finally, of the eight students at the Chilean university, seven consider MD to be a technical discipline and one considers it a scientific discipline. In sum, most future teachers assume in their discourse the technological character of MD, that is, they understand MD as science-oriented towards the improvement of the teaching and learning processes of mathematics.

Regarding the criteria with which future teachers base the improvement of the teaching and learning process of mathematics, it is evident that they implicitly consider the use of the six criteria of didactical suitability.

Regarding epistemic suitability, a future Ecuadorian professor argues that an improvement in the teaching and learning of mathematics consists in working on the richness component of mathematical processes, in particular, the problem-solving process, which is evidenced below:

Identify the situations (problem). Investigate the existence of similar (previous) problems, develop possible solutions and evaluate them, prepare proposals (Student 7, Ecuador).

Regarding cognitive suitability, future teachers considered the importance of working on students' prior knowledge. For them, the improvement consists of:

Being able to learn by teaching, because it is not enough just to fill out the blackboard and believe that the students learned, without at least knowing that the students already know something (Student 4, Brazil).

Diagnose the academic level of students, find the shortcomings they have had in previous years and establish flexible strategies. (Student 18, Ecuador).
Investigating where are the deficiencies in the learning of mathematics and being able to reinforce and also identify what type of content is relevant (Student 6, Chile).

In addition to having an overview related to prior knowledge, improvement consists of making curricular adaptations to individual differences, another component of cognitive suitability.

**Emphasize personalized teaching considering the virtues of children. (Student 17, Ecuador).**

They have also considered the importance of activating relevant cognitive processes, an indicator of the high cognitive demand component of the cognitive suitability criterion.

*It must be considered that as a teacher, students are not looking for the correct result, but rather that they learn to reason and that they arrive at the correct answer. That is, take into consideration the process and not just the product. (Student 22, Ecuador).*

Future teachers considered it important to consider the teacher-student interaction, a component of the interactional criterion:

*It means an advance in the communication processes between the parties involved in learning so that the process occurs effectively. (Student 15, Brazil).*

From an emotional point of view, future teachers declare that improving mathematics teaching and learning consists in promoting self-esteem, avoiding rejection, phobia, or fear of mathematics, which is related to the component called “emotions” of the emotional suitability criteria (Breda, Pino-Fan and Font, 2017):

An improvement means overcoming the prejudice about mathematics and the use of pedagogical and technological resources so that the student learns without trauma or fear and becomes a subject who learns with confidence. (Student 8, Brazil).

They also consider that it is important to propose situations of interest to students, having the “interests and needs” component of the criterion of interactional suitability implicit:

**Transmit mathematics in a clear and simple way so that students do not get bored, or lose interest, much less come to hate mathematics (Student 3, Ecuador).**

Related to the media, many students considered that in order to have an improvement in the teaching and learning of mathematics, it is important to use diverse and adequate resources a resource component of the mediational suitability criterion.

*It means a positive change in the strategies and resources that are used to teach mathematics (Student 7, Ecuador).*

**Depending on the age of the students and their needs, recreational activities can be carried out and specific material used. (Student 3, Ecuador).**

They also considered it important that the didactic proposals take into account the socio-labor utility, a component of the ecological suitability criterion:
I believe that when inserted in the reality of the student, in a more practical way it would improve the teaching of mathematics. (Teacher 9, Brazil).

Change the paradigm and the conception that we have of mathematics and its teaching. Working mathematics from a more contextualized perspective (...) the main thing that we find useful for what is learned in this subject in everyday life. (Student 2, Chile).

They also considered didactic innovation, one of the components of the ecological suitability criterion:

I think that the didactic proposals should be striking and innovative that seek or serve as support to the teacher to improve the learning development of students by capturing their attention. (Student 1, Ecuador).

Through new, more innovative methodologies, with concrete, symbolic and pictorial material. Leaving aside the traditional way of teaching mathematics, where it is learned or understood through memorization or meaningless mechanical learning (Student 4, Chile).

There was a future teacher, in particular, who in his statement has implicitly suggested that improving the teaching and learning of mathematics is related to the epistemic part, that is, the knowledge of the mathematical content to be taught; learning, in particular, formative evaluation processes and curricular adaptation and the media, in particular, the time necessary to work on certain content:

Take into account the content. Mastering the subject and content. Adapt to the environment. Analyze the way students learn. Visualize if there are special educational needs and make adaptations for them. Calculate a reasonable time for teaching the subject. Teach the class. Take assessments (ongoing at the end). With respect to evaluations, reinforce the topic taught. (Student 23, Ecuador).

Another relevant aspect emerges from the discourse of future teachers is that improving the teaching and learning of mathematics consists of an improvement in teacher training, in particular, in the strengthening of competence in didactic analysis of their practice:

An improvement in teacher training aimed at finding new paths in the teaching of mathematics in the classroom that it offers. (Student 7, Brazil).

It means seeking strategies and models for constant improvement, focused mainly on teacher training. In order to achieve the benefit for students (Student 5, Chile).

Starting from the observation of a mathematics class given in a certain educational unit, in order to know the advantages and disadvantages of this type of teaching. Carry out an analysis of the aspects that could be improved, put them into practice, and check their effectiveness as a didactic proposal. (Student 13, Ecuador).
In quantitative terms, it can be inferred that the majority of the surveyed teachers relate that the improvement of the teaching and learning processes of mathematics are related, to a greater extent to the criteria: cognitive, ecological, and emotional and, to a lesser extent, to the criteria: interactional, mediational and epistemic. That is, in the statements of future teachers, for example, very few comments were made about the improvement being related to the quality of mathematical knowledge that the future teacher should have (epistemic criterion), or to the process of classroom organization and management (interactional criteria). Likewise, they consider it very important to consider the context in which the student operates, didactic innovation (ecological criteria), and prior knowledge and evaluation methods (cognitive criteria).

<table>
<thead>
<tr>
<th>Perceived meanings that future teachers from Ecuador, Brazil and Chile attribute to MD</th>
</tr>
</thead>
<tbody>
<tr>
<td>MD is perceived to be a technological discipline</td>
</tr>
<tr>
<td>MD is perceived to improve the teaching and learning of mathematics</td>
</tr>
<tr>
<td>MD is perceived to be a scientific discipline</td>
</tr>
<tr>
<td>MD is perceived to be a tool or an innovative didactic in the teaching and learning of mathematics</td>
</tr>
</tbody>
</table>

Table 7: Cross-country analysis of future teachers’ perception of MD.

CONCLUSIONS

The objective of this work was to determine the meaning that future teachers attribute to the didactics of mathematics and the criteria that they base the improvement of the teaching and learning process of that discipline. The data allow us to infer that, regardless of the country and the characteristics of the training courses, the meanings attributed to MD are similar. On the one hand, in the three countries, the majority of future teachers consider MD to be a technological discipline whose role is to improve the teaching and learning of mathematics. On the other hand, few future teachers consider MD as a scientific discipline based on theoretical and methodological tools that serve to describe and explain the teaching and learning processes of mathematics. Furthermore, it could be inferred that future teachers present coherent arguments when explaining how the teaching and learning processes of mathematics could be improved. These arguments are mainly related to some current principles and trends in Mathematics Education and some results of research carried out in the field of Mathematics Didactics, as indicated by Breda, Font, and Pino-Fan (2018) and Esque and Breda (2021).

In this line, the present study coincides with the need to respond to the new paradigm of the professor-researcher (Nortes and Nortes, 2011; Oliveira and Fiorentini, 2018; Zumaeta, Fuster and Ocaña, 2018), so that the scientific and technological character of the MD can be recognized.
(Godino, Batanero and Font, 2019) in order to problematize their professional work and provide well-founded answers to improve teaching and learning processes.

Although the limitations of this qualitative study are related to a contingent of 49 future teachers, the criteria considered by them to generate an improvement in the teaching and learning processes of mathematics are interlinked. Above all, to the cognitive, ecological, and emotional criteria and, to a lesser extent, to the interactional criteria, mediational and epistemic. This study confirms findings of the previous research (Breda, 2020; Breda, Pino-Fan and Font, 2017; Giacomone, Godino and Beltrán-Pellicer, 2018; Hummes, Font and Breda, 2019; Morales-López and Font, 2017; 2019; Sánchez, Font, and Breda, 2021; Seckel, Breda, Sánchez and Font, 2019; Seckel and Font, 2015; 2020), where the implicit use of the didactical suitability criteria is observed by mathematics teachers (in this case, during their initial training).

Acknowledgments

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References


Modeling Through Model-Eliciting Activities: An Analysis of Models, Elements, And Strategies in High School. The Cases of Students with Different Level of Achievement

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Abstract: A mathematical client-driven task known as Model-Eliciting Activities was implemented with students of different levels of achievement (i.e., low, average, and high) at the high-school level. The study strived to prove that Model-Eliciting Activities can be solved by students at any achievement level and be used as an assessment tool. Students collaborated in teams of three to develop solutions that met the client’s needs. The model-solutions were compared and contrasted among several dimensions and achievement levels, considering the quality of the final product-solution based on the Quality Assessment Guide, the intermediate product composed of the type of models created, the strategies followed, and the elements of the mathematical construct. A Model & Modeling perspective was considered as a framework in which students’ teams comprised the cases studied and analyzed. Findings show that students were able to create and elaborate models-solutions regardless of the level of achievement, with a comparable quality but with different strategies. Although student’s solutions were similar in sophistication across achievement level, more studies are needed to evaluate high schoolers’ solutions to these types of mathematical task.

Keywords: Modeling, High School, Assessment, Achievement-level

INTRODUCTION

It is not uncommon for students across all levels of achievement to struggle when solving traditional textbook mathematical problems, but below-average students tend to struggle even more, often because their teachers underestimate their abilities (Madon et al., 1997). However, with the right approach, motivation, and engagement, students at any achievement level can solve problem activities beyond their teachers or anybody else’s expectations. In particular, the type of problem-solving activities that were used for this project are the ones known as though-revealing or Model-Eliciting Activities (MEA).
By using MEAs, teachers can help all students without leaving “low-achievers” behind, since these activities enable equally good outcomes across all achievement levels (i.e., low, average, and high). The main difference between MEAs and traditional classroom tasks (Alsup & Sprirler, 2003; Dewey, 1998) is that the design of MEAs leads the problem-solvers to focus on the process toward a solution (Lesh & Doerr, 2003) rather than looking for a single answer using predetermined mathematical algorithms.

That is to say, the solution to a MEA is the model the problem-solver produces—known as problem representation— to provide the best answer to a problem, not the answer itself. MEAs thus enable students to interpret, invent, and find solutions in ways that “jump” the barriers of “achievement stereotype” by focusing on process instead of more rigid strategies. For example, Carmona and Greenstein (2007) compared the solutions brought to the same MEA by a group of 3rd graders and a group of post-graduate math and science students. The activity required the problem-solvers (i.e., the students) to rank 12 teams of players presented on a two-dimensional coordinate system. Working in small groups, the students were required to model a solution to rank the top five teams with the most wins. Each group worked for approximately two hours developing a solution to the problem. By building on their previous knowledge and experiences, the problem-solvers needed to decide which solution-paths led to the best outcome, and which mathematical ideas worked best in developing their constructs.

Each group of students was very different in their respective mathematical skills, age, and grade level, yet Carmona and Greenstein showed that both the elementary and post-graduate students developed adequate solutions. As expected, the post-graduates’ solutions were more mathematically advanced (e.g., they used vector concepts and Pythagorean formulae) and better justified. However, the elementary students’ models were as adequate — in terms of the process and model developed — as the post-graduates’ for providing accurate answers to the proposed problem. If the elementary students are considered “low-achievers” and post-graduates as “high-achievers,” then we can conclude two things: first, that MEAs can work to the “level the playing field” across achievement levels, and I seek to verify and demonstrate that students at these different levels can indeed provide equally adequate solutions to activities like MEAs; second, that the degree of difficulty of any particular problem posed for each achievement level is “better determined by the solution the problem solver produced” (Carmona & Greenstein, p. 253) than by the ability (or lack thereof) teachers presuppose students to have attained. In contrast to Carmona and Greenstein, my research shows that different achievement levels within the same grade level work the same way: although low-achievers often use different processes than their average- or high-achieving peers, they attain equally adequate outcomes, and in some instances even better in terms of the justification of their ideas, the reasoning, and the explanation.

In the context of this research (i.e., high school), more and better strategies for teaching students how to solve real-life problems effectively, creatively, and mathematically are needed. Model-eliciting activities are presented here to illustrate a possible alternative for presenting
instructional models in ways that enhance and elicit mathematical literacy for high school students at different achievement levels, and to counter the stereotype that low- and (perhaps) average-achieving students cannot solve open-ended mathematics activities (like the one presented here) as adequately as students stereotyped as high-achieving can, and even to show they can “jump” the boundaries that define these stereotypes. To this end, I am answering the following question: To what extent the quality of the student’s final and intermediate product solutions (i.e., mathematical strategies, type of models, and elements) varies both within and across the achievement levels?

**Theoretical Framework**

Solving real-life problems in the modern world is neither about following a certain set of rules nor working in isolation to find an answer. People today frequently need to adapt problem-solving strategies to unique and complex situations and to develop collaborative solutions to difficult problems (Lombardi, 2007). However, when students in school are taught to solve problems, they are often presented with unrealistic situations (e.g., textbook problems) that require them to follow particular rules or strategies (already practiced and mastered) to solve a problem (usually individually) that is seldom similar to any real-life situation. Furthermore, the concepts they are tasked to learn are limited to grasping and/or memorizing a set of rules and passing standardized tests about topics that are usually quickly forgotten. In mathematics classrooms, in particular, it is common to find this type of traditional problem-solving perspective — learning to follow specific steps in order to solve formulaic problems. However, this rarely helps students in solving real-life problems that require a level of abstraction. By contrast, in this research project, I took an approach in which students can work collaboratively to interpret situations in multiple ways, evaluate possible solution paths, and enter into a cycle of description, explanation, and prediction as they solve a problem; and in which the development of a useful and powerful solution is beneficial not only to students who excel in memorizing mathematical solution-steps or have attained a high level of achievement in school, but to everyone. This approach is called the Model and Modeling perspective (MM) (Lesh & Doerr, 2003).

Within the MM perspective, models are conceptual tools used to mathematize a real situation and modeling is the process in which a model is adapted or constructed to provide a solution to a problem (Lesh & Doerr, 2003; Zandieh & Rasmussen, 2010; Lesh & Lehrer, R., 2003). Following MM, the model-construction process is an interactive leaning cycle in which students collaboratively (1) examine a situation, (2) identify the problem to be solved and the variables involved, (3) formulate a model or a problem representation, (4) test the model, (5) interpret the results, (6) validate their model solution’s applicability to the original situation, and (7) apply the model to other similar situations to test its usefulness (Kang & Noh, 2012). Moreover, since MM emphasizes teamwork when solving a problem, students not only reflect on their own thoughts individually, but also communicate their ideas in ways that other team members can...
evaluate, reject, or accept. In general, each member learns from the different perspectives that emerge through collaboration (Lesh et al., 2003).

The MM perspective supports and advocates unstructured, collaborative learning, since learning occurs in the social context (Vygotsky, 1962) of students’ interactions, when team members mutually “negotiate goals, define problems, develop procedures, and produce socially constructed knowledge in small groups” (Springer et al., 1999, p. 24). Many theories and theorists have addressed the benefits of learning in small groups (Springer et al., 1999): Piaget (1926) and Vygotsky (1978) in cognitive psychology, Deutsch (1949) and Lewin (1935) in social psychology, Dewey (1943) in experiential education, and Belenky et al. (1986) in humanist and feminist theory. However, the collaborative learning in the MM perspective is primarily rooted in the developmental theories of Piaget (1926) and Vygotsky (1978) in which “face-to-face work on open-ended tasks—projects with several possible paths leading to multiple acceptable solutions—facilitate cognitive growth” (Springer et al., 1999, p. 25). Based on this principle, MM recognizes that (1) it is crucial for students not simply to argue and discuss their opinions, but to share each other’s ideas and perspectives when working collaboratively — i.e., an idea similar to the multiple-perspective principle of Lesh et al. (2003) and (2) that students can uncover their inadequate reasonings as disagreements arise during their discussions, and that working out these disagreements enhances the understanding of all (Springer et al., 1999).

MM thus proposes types of problem-solving activities that furnishes to the learners a space to collaborate with others, to try out, reflect on, and re-enact their own and others’ ideas — and MEAs are such an activity. In MEAs, the social and communal construction of knowledge takes place (Vygotsky, 1978; Tangney, FitzGibbon, Savage, Mehan, & Holmes, 2001) and “the learners not only construct their own knowledge while interacting with their environment [and other people] but are also actively engaged in the process of constructing knowledge for their learning community” (e.g., while working in teams) (Tangney et al., 2001, p. 3114). In collaborative groups, individuals merge their knowledge to strengthen and broaden their skills while achieving a common goal. This escalates their motivation toward an interest in problem-solving (Zawojewski, et al., 2003). It also increases the possibility for students to create and invent more sophisticated and powerful solutions using mathematical representations (e.g., artifacts and constructs) and inscriptions — e.g., graphs, tables, diagrams — to mediate their thought processes and reasoning.

Model, Modeling and Model-Eliciting Activities

In a classroom context, models are a “student-generated way of organizing their mathematical activity with physical and mental tools” (Zandieh & Rasmussen, 2010, p.68). Moreover, Lesh and Doerr (2003) defined models as: “Conceptual systems (consisting of elements, relations, operations, and rules governing interactions) …used to construct, describe, or
explain the behaviors of other system(s)” (p.10). Modeling, then, is the process through which students construct or adapt and structure conceptual systems — i.e., models — in order to solve real-life problems (Zandieh & Rasmussen, 2010; Ekmekci, 2013).

MEAs are open-ended client-driven problem-solving activities in which students working in small teams are encouraged to develop and generate useful solutions (i.e., conceptual systems or models) for a “client”. These solutions are required to be reported to the client in a letter-format way, where students provide a detailed explanation of their model-solution. Students generate solutions by repeatedly communicating, testing, refining, and extending their thoughts (Lesh et al., 2000). These thought processes often involve several modeling cycles. Within MEAs, a modeling cycle is the process by which students describe, manipulate, predict, and verify their mathematical constructs, then adapt, modify, and/or refine their own knowledge and ideas (Lesh & Doerr, 2003). Often, they must go through several cycles of modeling to interpret and improve their products (Kaput, 1998) in ways that go beyond “just providing an answer” to offering unique solutions to a problem (Lesh & Doerr, 2003). Multiple cycles of modeling enhance students’ learning because students “have multiple opportunities to invent, revise, and then compare the explanatory adequacy of different models” (Lehrer & Schauble, 2006, p. 382).

**Setting and Data Collection**

The target population for this study was 11th grade high school students at different achievement levels at a private school located in northeastern Mexico. The 11th grade cohort contained approximately 74 students divided into three sections of 22 to 28. To form the teams, the classroom teacher and I (as a co-teacher) asked the students to form teams of two or three members without forcing or imposing any type of categorization, randomization, or selection rule (i.e., students formed these groups in a spontaneous way). These naturally formed teams represent the cases and focus of this study and analysis. Later, the classroom teacher categorized each team as low-, average, and high-achievement based on the students’ individual performance in class (as measured by their grades, test scores, and classroom activities). In the end, 24 teams were formed: 14 average-, 5 low-, and 5 high-achievement.

The fieldwork for this study was divided into three phases. In the first week, I implemented *The Team Ranking Problem* (Greenstein et al., 2008), the following week, *The Hybrid vs. Gas Car* problem (Elliott, 2014), and two to three weeks later *The Historic Hotel* problem (Aliprantis & Carmona, 2003). Students spent approximately between 150-180 min. solving each MEA. For all the three MEAs, I had three information sources: I gathered all the students’ paperwork and artifacts, observed and took notes during the project time, and video-recorded the students’ interactions, presentations. Students’ paperwork and artifacts included all the work the students did individually and in teams when developing their models, and the letter to the “client” each team wrote explaining and detailing their solution or mathematical construct. Observations included notes about the students’ mathematical ideas taken by me and the classroom teacher. For
the purpose of this manuscript, only solutions for the last MEA implemented is reported (i.e., *The Historic Hotel Problem*).

**The Activity**

*The Historic Hotel Problem is an* MEA developed by Aliprantis and Carmona (2003). It is based on an economics problem (Aliprantis, 1999) that asks and encourages students to develop a mathematical model:

Mr. Frank Graham has just inherited a historic hotel. He would like to keep the hotel, but he has little experience in hotel management. The hotel has 80 rooms, and Mr. Graham was told by the previous owner that all of the rooms are occupied when the daily rate is $60 per room. He was also told that for every dollar increase in the daily $60 rate, one less room is rented. So, for example, if he charged $61 dollars per room, only 79 rooms would be occupied. If he charged $62, only 78 rooms would be occupied. Each occupied room has a $4 cost for service and maintenance per day.

Mr. Graham would like to know how much he should charge per room in order to maximize his profit and what his profit would be at that rate. Also, he would like to have a procedure for finding the daily rate that would maximize his profit in the future even if the hotel prices and the maintenance costs change. Write a letter to Mr. Graham telling him what price to charge for the rooms to maximize his profit and include your procedure for him to use in the future.

The activity has been validated and tested in many different educational settings, with different participants. For example, Aliprantis and Carmona (2003) implemented it in middle schools, Dominguez (2010) also applied it in a calculus class in a southwest Texas university, and Ekmekci (2013) implemented it when working with pre-service mathematics and science teachers. Though implemented in many different contexts, this MEA has not been implemented in a high school or with the intention of comparing and contrasting the model-solutions of low, average, and high-achieving students. The mathematical ideas this activity addresses cover the topics of patterns, variables, and parabola (i.e., quadratic function).

**Methodology**

The study was conceived as a series of case studies, which were then compared and contrasted along various dimensions (Thomas, 2011; Goodrick, 2014; Hayes, 2000). Case studies are especially well-suited to my purposes here because they allowed me to observe both single individuals and participant groups and to study the students’ solving processes as they unfolded,
with a higher level of detail than if I were examining a larger population (McLeod, 2008). Case study is an empirical research method that “investigates a contemporary phenomenon within its real-life context...in which multiple sources or evidence are used” (Yin, 2003). In this research, I adopted Creswell et al., definition of case studies as “a qualitative approach in which the investigator explores a case or multiple cases over time through detailed, in-depth data collection involving multiple sources of information” (2007, p. 245). Although I collected audio of all teams, video of some selected teams, only the written work students generated for the last MEA implemented (i.e., The Historic Hotel Problem) is considered in this report.

To evaluate the quality of the student’s final solutions I analyzed the students’ written report to the client (i.e., the letter to the client) following the Quality Assessment Guide (QAG). In the letter to the client, students are required to explain in detail their solutions. The QAG (See Table 1) is an instrument Lesh and Clarke (2000) developed to assess the quality of solutions students had produced in a specific MEA. The QAG rates the quality of a solution considering five levels of performance, i.e., those that: (1) require redirection, (2) require major extensions or refinements, (3) require only minor editing (4) are useful for the specific data given and (5) are sharable or reusable. All these performance levels are based on the client’s needs.

**Table 1**

*Quality Assessment Guide*

<table>
<thead>
<tr>
<th>Performance level</th>
<th>How useful is the product?</th>
<th>What might the client say?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Requires Redirection</td>
<td>The product is on the wrong track. Working longer or harder won’t work. The students may require some additional feedback from the teacher.</td>
<td>“Start over. This won’t work. Think about it differently. Use different ideas or procedures.”</td>
</tr>
<tr>
<td>Requires Major Extensions or Refinements</td>
<td>The product is a good start toward meeting the client’s needs, but a lot more work is needed to respond to all of the issues.</td>
<td>“You’re on the right track, but this still needs a lot more work before it’ll be in a form that’s useful.”</td>
</tr>
<tr>
<td>Requires Only Minor Editing</td>
<td>The product is nearly ready to be used. It still needs a few small modifications, additions, or refinements.</td>
<td>“This is close to what I need. You just need to add or change a few small things.”</td>
</tr>
</tbody>
</table>
To analyze students’ intermediate-products, I considered several characteristics of the student’s work (Dominguez, 2010; English, 2010; Aliprantis & Carmona, 2003; Ekmekci, 2013; Greenstein & Carmona, 2013):

a. The type of model-solution (Lesh & Zawojewski, 2007) created,

b. The strategies (Kent et al., 2015) followed to obtain their model, and

c. The components (Lohse, Biolsi, Walker, & Rueter, 1994) — also called elements of the visual representation — used to represent the mathematical construct and data.

To deeply analyze and study the intermediate-products, and for the purpose of this report, I considered only the last MEA implemented, The Historic Hotel. All 24 teams’ model-intermediate-products solutions were coded based on the characteristics of their model-solutions.

Findings

The quality of the student’s final solution, which is based on the student’s report or letter to the client, was rated using the QAG as framework. The 24 teams’ final products for the Historic Hotel MEAs were analyzed by two reviewers with experience in mathematics education at different grade levels. The reviewers scored the students’ final model-products based on the levels of performance of the QAG. The inter-rater agreement score, which represents the percentage of agreement between raters (Tinsley & Weiss, 2000; Gwet, 2014), was approximately 88%. After discussing and resolving the differences, the inter-rater agreement score ended in 92%. The QAG scores for each team are shown in the following table.
Table 2

Teams’ Quality Assessment Scores and Means

<table>
<thead>
<tr>
<th>Team</th>
<th>Level of Achievement</th>
<th>Historic Hotel MEA’s Score</th>
<th>MEAN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guardianes de la Galaxia</td>
<td>Low</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Dinamita</td>
<td>Low</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>VACAOS</td>
<td>Low</td>
<td>3</td>
<td>3.21</td>
</tr>
<tr>
<td>LGH</td>
<td>Low</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>LNA</td>
<td>Low</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Mateatletas</td>
<td>High</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Carrojan</td>
<td>High</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Kam Girls</td>
<td>High</td>
<td>3</td>
<td>3.40</td>
</tr>
<tr>
<td>SEJUSA</td>
<td>High</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>MA²³</td>
<td>High</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Princesas</td>
<td>Average</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Minions</td>
<td>Average</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Los Compadres</td>
<td>Average</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>APS</td>
<td>Average</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>CEZAMO</td>
<td>Average</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Chetos</td>
<td>Average</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>FC Valle</td>
<td>Average</td>
<td>4</td>
<td>2.64</td>
</tr>
<tr>
<td>Aguilas Doradas</td>
<td>Average</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>CFN</td>
<td>Average</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>OP</td>
<td>Average</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Fleurs</td>
<td>Average</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>ICA</td>
<td>Average</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Enchiladas</td>
<td>Average</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>GYF</td>
<td>Average</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

The QAG is a general instructional approach to scoring teams’ work (Hjalmarson et al., 2011; Lesh & Clarke, 2000). It only considers the written reports and/or presentations the students created and generated as results of their model-construction process (Clement, 2008). As a rubric that scores the quality of the students’ work, it provides an overall idea on how the students’ model solves the client’s problem and, at some point, a way to compare the final products based on the client’s needs. For example, Table 2 shows that most students, regardless of their performance level, had a score of 2 or 3, meaning that they might need either more time to work on the problem or more practice solving MEAs. In addition, note that low-achieving teams had a higher score mean, when compared to their average peers, meaning that they developed better solutions for the
client based on the QAG, and a very similar mean in contrast to their high-performing peers. The QAG is a fairly simple way to understand the teams’ final solutions in a very typical manner, by scoring and providing a grade. But it fails to consider a deeper analysis of the type of model the students constructed, the mathematical components/elements they considered for their models, and the types of strategies they used in developing their models. In the end, the QAG does not provide full evidence of the richness of the students’ solutions. It is not possible to properly define how good a solution is just by looking at a number from one to five. Certainly, a deeper way to characterize, compare, and contrast the teams’ work is required, rather than simply considering the QAG as a unique method of evaluation and analysis of the students’ models-solutions. Therefore, in the next section I analyze the intermediate product-solutions that students generated aiming to have a deeper understanding of their models.

**Intermedia Product-solutions**

The intermedia product solution refers to the models and strategies developed by teams as they solved the activity, and the components and elements they used to create their models. After analyzing the student’s data, their models were generalized in different types. Likewise, the elements or components of the mathematical construct were also categorized.

**Model-Solution types**

I have categorized all 24 teams’ solutions for *The Historic Hotel* into five general forms of models the students created. In a general sense, most solutions considered the profit — “Ganancia” in Spanish (G) —, maintenance cost (SMC) —”Mantenimiento” —, the booked rooms (#Rooms), and the room’s price ($Price).

In the first model, the profit is the result of subtracting the maintenance cost of the rooms from the cost of all booking rooms. Two subcategories emerged from this model-type solution: one that considers providing maintenance only to booked rooms (1A), and another that considers providing maintenance service to all rooms, regardless of whether maintenance is needed (1B). In this second subcategory, the maintenance cost is the result of multiplying the daily cost of maintenance ($DCM) — MX$40 — by the total number of rooms (80):

1A: $G= (#\text{ Rooms} \cdot $\text{Price}) – (#\text{ Rooms} \cdot $\text{DCM})$
1B: $G= (#\text{ Rooms} \cdot $\text{Price}) – \text{SMC} \quad \text{SMC=80} \cdot 40 = \text{MX$3200}$

The second model-type is a factored version (2A) of the original profit equation 1A in which the variable #Rooms is a common factor in all terms. In addition, another subcategory-model emerged that considers a set of alternate models that model the number of rooms to be booked and the maintenance cost (2B). In the former model the initial booking price (SIBP) — MX$600 — and the increased desired price (SIDP) are considered main variables to determine
the number of booked rooms (#Rooms). Furthermore, a variable “x” represents any increment in the maintenance cost:

2A. \( G = (\text{Price} - \text{DCM}) \times (\# \text{Rooms}) \)
2B. \( G = (\text{Price} - \text{DCM}) \times (\# \text{Rooms}) \), where
#Rooms = 80 – [ ($IDP - $IBP) / 10 ]
SDCM = 40 + x

The third model-type that emerged in the students’ work considers both the number of booked (#Rooms) and unbooked rooms (#UBRooms) as a strategy to determine the increase in the initial booking price ($IBP):

3. \( G = (\# \text{Rooms}) \times [ ($IBP + (\# \text{UBRooms} \times 10) ] - (\# \text{Rooms} \times \text{SDCM}) \)

The fourth model-solution, which is the least sophisticated model in comparison with the other three models-type showed above, does not take into account the cost of maintenance. The rationale of the teams that decided this solution-path was that the maintenance cost would be included in the booking price, so it was not necessary to include it in the model. The model is only composed of the number of booked rooms and the booking price:

4. \( G = (\# \text{Rooms} \times \text{Price}) \)

Finally, the fifth category clusters all models of category 1 of the QAG. These require major reconsiderations, which may reveal a lack understanding of the problem statement of the MEA. Few teams fall into this category, however, and none of the “low-achieving” teams fail to propose a model-solution showing a profit. In fact, only three teams fall into this category: two “average,” and one “high-performing.”

In Table 3, I show the type of model that each team in the different achievement level created. Teams created different model types, and in many cases the type of model is the same regardless of the level of achieving. From Table 3, I made a rough count to obtain percentages of the different types of models the teams created with the intention of knowing how many model-types fell into each category. For example, I can demonstrate that 46% of the teams constructed the same type of solution (1A) to model the hotel’s profit \( G = (\#\text{Rooms} \times \text{Price}) - (\#\text{Rooms} \times \text{DCM}) \). From this percentage, 55% belong to the average-achieving teams, and 27% and 18% to the low- and high-achieving teams, respectively. In addition, 13% of the teams used a reduced factored version (2A) to model the profit, and only 8% considered providing service to all the rooms — model 1B.

Comparing solutions among teams, 60% of low-achieving teams created a model that fell into the 1A category, which is the most common model created among all teams. Most (80%)}
considered following strategy A to model their solution, and all or almost all of them used lists, charts, and tables as components to represent their mathematical construct. In contrast, only 40% of high-achieving teams used the model-type A, and another 40% used the 2B model, which is the more mathematically sophisticated model as it considers an extra set of models to determine the number of rooms to be booked and the maintenance cost. For the average-achieving teams, less than half created a model-type A, 20% used the factored model-version 2A, and a small percentage fell under the model-category type 5, in which the model needs to be restructured and reconsidered.

Model-Strategies

To obtain their models, the teams needed to evaluate the strategy they would follow considering all the characteristics and information provided in The Historic Hotel problem statement. In this section, I describe the three types of strategies the teams followed when developing their model-solution.

The first strategy-type (A) implies systematically increasing the initial price by 10, and decreasing the number of booked rooms by 1. For example, if the initial price of MX$600 for a booking room is increased by MX$10, then only 79 rooms get booked, rather than the 80 originally booked. The second strategy (B) considers increasing the initial price by MX$5—instead of by the MX$10 stated in the problem statement and decreasing the number of booked rooms by one only when the initial price had been increased by a multiple of 10. This strategy requires increasing the price more times than other strategies and booking fewer rooms, but it generates a higher profit in the long term. For example, increasing the initial price from MX$600 to MX$605, would still generate the profit of 80 booked rooms. However, increasing the price to MX$610 would reduce the number of booked rooms to 79. In the third strategy-type (C), the students considered systematically increasing the initial price by MX$100, and decreasing the number of booked rooms by 10. One more general strategy (D) includes the combination of any of the above strategies, mixing either the first and second, the first and third, or the second and third. In Table 3, I show the strategies used by each team, which fell into one of the three different strategies, and in some cases a combination of any of the strategy’s types.

Just as I did with the model-types, I made a rough count of the type of strategies that teams used in order to know how many strategies fell into each category. For example, just as with the low-achieving teams, 80% of high-achieving teams considered following strategy A. Similarly, almost 80% percent of the average-achieving teams considered also following strategy A. Only 13% of teams considered this strategy or a combination with any other strategy. Finally, strategy C was only considered by 4% of the teams. Most of the teams, regardless of achievement level, considered using the same type of strategy, which once again shows that MEAs might serve to level the playing field.
Components/Elements of the mathematical constructs

In the definition of models I adopted in this research project, conceptual tools (models) are composed of elements that allow the mathematization of a real-life situation with the ultimate purpose of predicting how that real-life situation would behave (Lesh & Doerr, 2003; Blumm & Ferri, 2009; Mousoulides, 2011; Blomhøj, 2004). For such predictions, the components or elements of the model and the strategies become essential in helping to make connections from real life to the mathematics world and vice versa, to visually represent the collected data, and ultimately predict a situation. While working on their model-product-solution, students created and used different types of components that helped them to reach a better understanding of the situation they were modeling—in this case the maximization of a hotel’s profit. The components of the mathematical constructs that emerged in the students’ work were: charts (C), tables (T), lists (L), and graphs (G). Examples of how the students used these types of components are given below, but to provide a basic idea here, I present some examples in Figure 1:

Figure 1

Mathematical components of the written solution

![Figure 1](image)

Figure 1. The figures above are examples of the types of components created and used during the process of model-development.

Similar to what I did before with the models and strategy types, I also made a rough count of the elements teams decided to create and use when developing their solution. As can be seen in the summary Table 3, 80% of the low-achieving teams used graphs (G) as a type of element to represent their data and mathematical constructs, and most of them used lists (L) and tables (T). Similarly, all the high-achieving teams used graphs (G) and most of them considered using lists (L) and tables (T). In contrast, few average-achieving teams considered creating graphs or charts, instead more than half used tables or lists as a way to represent their constructs and data.
I have detailed and explained above the three characteristics of the students’ intermediate products — i.e., the model, the strategies, and the components/elements — based on what they showed in their written reports, worksheets, artifacts, and final presentations. In Table 3, I show a summary of the models, strategies, and components each team created and used during their model-solving process.

Table 3

Summary of the model-solution characteristics by team

<table>
<thead>
<tr>
<th>Team</th>
<th>Level of achievement</th>
<th>Type of Model</th>
<th>Type of Strategy</th>
<th>Mathematical Elements used/created</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guardianes de la Galaxia</td>
<td>Low</td>
<td>3</td>
<td>A</td>
<td>G,Ch,L</td>
</tr>
<tr>
<td>Dinamita</td>
<td>Low</td>
<td>1A</td>
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<td>L,T,Ch</td>
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<td>1B</td>
<td>A</td>
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<tr>
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<td>G</td>
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<tr>
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<td>B</td>
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<td>A</td>
<td>T,L</td>
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<td>SEJUSA</td>
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<td>2B</td>
<td>A</td>
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<td>High</td>
<td>2B</td>
<td>A</td>
<td>G,T,L</td>
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<tr>
<td>Princesas</td>
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<td>3</td>
<td>A</td>
<td>T</td>
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<td>Minions</td>
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<td>A</td>
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<td>Los Compadres</td>
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<td>APS</td>
<td>Average</td>
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<td>C</td>
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<td>1A</td>
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<td>A</td>
<td>T</td>
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<td>Average</td>
<td>1A</td>
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<td>OP</td>
<td>Average</td>
<td>1A</td>
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<td>G,T</td>
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<td>1B</td>
<td>A</td>
<td>T</td>
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<td>ICA</td>
<td>Average</td>
<td>2A</td>
<td>A</td>
<td>T,L</td>
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<td>Average</td>
<td>2A</td>
<td>A</td>
<td>G,L</td>
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<tr>
<td>GYF</td>
<td>Average</td>
<td>2A</td>
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</table>

Teams at different achievement levels considered similar models, strategies, and components to represent their models. The summary of the intermediate products (Table 3) is useful to get a better idea, in comparison to only considering the QAG, of how students construct their model-solutions and as a measure of comparison among all 24 teams. Only high-achieving
teams created, used, and developed solutions that required using more than a single model (e.g., model 2B), but in some aspects (e.g., strategies and components/elements), high-achieving teams’ final and intermediate solutions were similar across performance levels. Analyzing the teams’ products provides a general perspective on how comparable these solutions are.

Conclusion

High school students worked in teams of two or three members to solve open-ended mathematics activities known as Model-Eliciting Activities (Lesh & Doerr, 2003; Lesh et al., 1999). As they engaged in these activities, they “stretched and developed their conceptual understanding” (Capraro et al., 2007, p. 124), and viewed the emerging path-solutions from many different perspectives (Domínguez, 2010; Greenstein & Carmona, 2007).

The Achievement level as considered and established in this research project does not effectively determine the ability of students to generate solutions to open-ended mathematics activities like MEAs, which require complex thinking (Iversen & Larson, 2006; Lesh & Sriraman, 2005; Carmona & Greenstein, 2010). However, it was noticeable that some but not all high-achieving teams developed solutions that were more sophisticated, mathematically speaking, than those of the low-and average-achieving teams, based upon the Quality Assessment Guide, and the analysis of the intermedia products. Therefore, although the open-ended nature of the MEAs allows students to consider different solution paths for the same problem, much more evidence would be needed to show that MEAs could indeed serve to level the playing field among students of different achievement levels.

Many aspects of modeling research still need to be investigated further, particularly in the high school level, including student’s perceptions and belief of using open-ended task like the ones used here. Although—from the perspective of the teachers and myself as a co-teacher and researcher—students seemed to enjoy the collaborative work and the opportunities to externalize their thoughts, these was not considered in the current study.

The research I have presented here is a first step in empowering low-achiever students in mathematics, too often sent to remedial classes, and in breaking the stereotype of them as being unable to develop solutions to mathematical activities like MEAs as creatively, adequately, and powerfully as average- or high-achievement students. In fact, this study contributes to breaking the stereotype of high-achiever students always performing outstandingly well.

References


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Mathematical Problem-Solving in two Teachers’ Knowledge Models: A Critical Analysis

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Abstract: Two of the teachers’ knowledge models most widely used in the literature are the Mathematical Knowledge for Teaching (MKT) and the Knowledge Quartet (KQ). We develop an analysis of the limitations of the knowledge required for teaching problem-solving published during 1990-2018 which includes these models. This analysis revealed that MKT takes neither the nature of the process nor the knowledge accumulated by problem-solving research into consideration. While the KQ is subject to similar omissions, its major drawback is element overlap. We conclude that the knowledge required to teach problem-solving is not clearly envisaged in the theoretical teachers’ knowledge models analysed.

Introduction

Problem-solving (PS) is one of the fundamentals of classroom mathematics curricula (NCTM, 2000). As the parties responsible for delivering that curriculum, teachers must be more than mere competent solvers of the problems used in the lessons taught. Teachers’ ability to solve complex, cognitively demanding problems does not suffice to guarantee appropriate PS instruction (Lester, 2013). They must also have specific PS knowledge (Chapman, 2015; Piñeiro, 2019), a concept that has arisen from reflection and previous research (e.g., Weber & Leikin, 2016). For instance, the problems selected to teach mathematics and how they are posed in the classroom are influenced by teachers’ own understanding of the mathematical content involved, the educational aims pursued and their beliefs around mathematics, its instruction and their students’ capacities (Weber & Leikin, 2016). That state of affairs determines the need to elucidate the factors of PS not associated with teachers’ PS skills but that should form part of mathematics teachers’ acquis (Lester, 2013).

In light of the slow progress made in the field, research linking mathematics teachers’ knowledge to PS has been identified as an area in need of attention (Weber & Leikin, 2016). The studies conducted to date focus primarily on teachers as problem solvers, with a paucity of papers addressing PS from the perspective of their knowledge (Lester, 2013). Earlier research along these
lines shows that pre-service trainees’ and in-service primary education teachers’ limited knowledge of PS impacts their students’ PS proficiency (e.g., Depaepe et al., 2010).

Shulman (1986), commonly regarded to have laid the grounds for this area of research, theorised that teachers’ knowledge is characterised by seven dimensions. The element of his theory with the greatest impact on the subsequent research is the characterisation of a special type of knowledge specific to teachers that enables them to teach: the pedagogical knowledge of the content. The need for more specific dimensions have prompted researchers to re-interpret his model, however. Rowland, Huckstep and Thwaites’s (2005) Knowledge Quartet; Davis and Simmt’s (2006) Teachers’ Mathematics-for-Teaching; the Michigan Group’s (Ball et al., 2008) MKT; among others, are models that build on Shulman’s theories. All have focused primarily on the two domains of teachers’ knowledge highlighted in his articles: knowledge of content and pedagogical knowledge of content.

Teachers’ knowledge models are “framed predominantly around mathematical concepts” (Foster, et al., 2014, p. 98) which, as some researchers contend, prompts significant omissions in the role of processes such as PS. Lin & Rowland (2016) note:

Papers presented at PME include a number of proposals for the elaboration, or modification, of extant theories of mathematics teacher knowledge... While such studies usually add to acronym-overload in the field, some draw attention to gaps or conflicts in the mainstream teacher knowledge discourse. Both Chapman (2012) and Foster, Wake and Swan (2014) take up a critique that Shulman’s framework and its derivatives focus on knowledge of mathematical concepts at the expense of PS proficiency (p. 489).

Chapman’s and Foster et al.’s alerts constitute the basis of the work presented here, in the sense of shedding light on the differences between knowledge about mathematical' concepts and processes.

We differentiate processes and concepts in the sense of NCTM (2000). From that perspective the process aims to find solutions for “something or some situation [which] is a problem only when someone experiences a state of problematicity, takes on the task of making sense of the situation, and engages in some sense-making activity” (Mason, 2016, p. 263). The inference is that knowledge of mathematical problems cannot be positioned in any dimension unrelated to the solver, whereas it is normally positioned in the knowledge of content dimension, from which students’ role is absent. The previous example shows that, unlike concepts, which are normally associated with mathematical structure, representation, and contexts or modes of use (e.g., Castro-Rodriguez et al., 2016), processes are entities in themselves. Therefore, one of the perspectives that can be adopted is that there is a difference between concepts (knowing) and processes (doing) (NCTM, 2000). Therefore, the notions related to the processes are not necessarily mathematical knowledge about some specific concept. Thus, given that the knowledge models most widely used in mathematics education may not capture elements inherent in the nature of processes, we posed the following research question: How do the most widely used teachers’ knowledge models
address PS-related knowledge? We approach this issue from a theoretical perspective. First we identified knowledge required for teaching PS. We then reviewed the literature to select the teachers’ knowledge models cited most extensively by the community of researchers. The third step consisted in identifying the limitations related to PS in the to the two most often cited theoretical models. The present description of those steps is followed by a discussion of the implications of our analysis for the research and teacher training.

Knowledge for teaching PS: Theoretical perspective

The specific knowledge required for teaching PS can only be accurately identified if broaching not only from the solver’s perspective but in terms of the theoretical particulars of the PS process, which in turn calls for a clear understanding of what PS involves. From that perspective PS instruction calls for different types of knowledge. Chapman (2015) proposed a specific framework or theoretical model on the grounds of a review of the literature from 1922 to 2013. She noted that PS is not organised around the same categories as proposed in other teachers’ knowledge models. In her model teachers’ PS skill is deemed a primary asset on which knowledge for teaching builds, described as a complex network of interdependent types of knowledge. The model components are summarised in Table 1.

Table 1. Components of MPSKT (Chapman, 2016, p. 141)

<table>
<thead>
<tr>
<th>Knowledge of:</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical PS proficiency</td>
<td>Understanding what is needed for successful mathematical PS</td>
</tr>
<tr>
<td>Mathematical problems</td>
<td>Understanding of the nature of meaningful problems; structure and purpose of different types of problems; impact of problem characteristics on learners</td>
</tr>
<tr>
<td>Math PS</td>
<td>Being proficient in PS</td>
</tr>
<tr>
<td>Problem posing</td>
<td>Understanding of problem posing before, during and after PS</td>
</tr>
<tr>
<td>Students as mathematical problem solvers</td>
<td>Understanding what a student knows, can do, and is willing to do (e.g. students’ difficulties with PS; characteristics of good problem solvers; students’ PS thinking)</td>
</tr>
</tbody>
</table>
Table 1. Components of MPSKT (Chapman, 2016, p. 141)

<table>
<thead>
<tr>
<th>Knowledge of:</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instructional practices for PS</td>
<td>Understanding how and what it means to help students to become better problem solvers (e.g. instructional techniques for heuristics/strategies, metacognition, use of technology, and assessment of students’ PS progress; when and how to intervene during students’ PS).</td>
</tr>
<tr>
<td>Affective factors and beliefs</td>
<td>Understanding nature and impact of productive and unproductive affective factors and beliefs on learning and teaching PS and teaching</td>
</tr>
</tbody>
</table>

Further to Chapman’s (2015) theoretical model, from the perspective of PS as professional knowledge, the issues posed around problems include understanding what a problem is, what PS is and what learning and teaching PS involves.

**Teaching PS: What knowledge does it require from teachers?**

In order to frame the authors’ answer, we discussed in the following three sub-sections, will subsequently serve to analyse the teachers’ knowledge models addressed.

**Mathematical problems, their solution and teachers’ knowledge**

Professional PS knowledge entails a command of problems per se and problem types. The classroom use of different problems calls for specific knowledge of the possible types that can be defined (routine or non-routine, for instance). Capitalising on the potential of a problem necessitates a knowledge of the mathematical complexities involved in the problem and its solution. Consequently, teachers must understand problem types and their properties. While none of the several classifications of problems in place has merited full consensus, researchers concur on the acceptability of certain dichotomies, such as applied/non-applied, routine/non-routine or open/closed.

Like all other problems, routine non-applied problems, also called exercises, require teachers to have mathematical knowledge and of their students. In our definition of problem, the learner is unaware of the pathway to solve it. Therefore, if an exercise of the type (27x5)-18 = □ is to be deemed a problem, the student must be unfamiliar with some step in the algorithms involved or with the use of parentheses. Otherwise, the exercise would not constitute a problem. In other words, in this type of tasks problem conceptualisation necessitates teachers’ knowledge of content and of their students.
An applied routine problem of the type “Maria has 12 apples that she wants to set out on plates. Each plate must have 3 apples. How many plates does she need?” requires teachers to know various multiplicative arithmetic structures and the strategies students might use to solve the problem. At the same time, they must know how the problem variables may interfere with one another, creating difficulties that would translate into student error. Errors might arise around the complexity of the multiplication, for instance, and the roles of dividend and divisor in asymmetrical problems; around language, relative to the position of the unknown in the wording or data sequencing; or inversion.

Non-routine non-applied problems require teachers to have profound mathematical knowledge. Although their use has been confined to entertainment or brain-teasing, they can be vehicles for developing independent mathematical thinking. They therefore call for problem-solving knowledge that enables teachers to prompt discussion of and verify predictions or conjectures.

Applied non-routine problems, whether closed as in “how many squares are there on a chessboard?” or open as in “how much paper is used in your school in a week?” are normally deemed to have greatest potential for developing mathematical skill (Lester & Cai, 2016). Such problems require a much more complex spectrum of knowledge, focused less on mathematical concepts and more on the solving process. Teachers must provide opportunities to generate and discuss strategies, help learners apply mathematical knowledge and induce discussion of both mathematical knowledge and the assessment of the strategies used. Such problems also require teachers to express beliefs and conceptions that favour the use of the problem to its full potential: they must believe that the problem may be solved via different valid pathways, have several acceptable answers or no answer at all. In problems of this nature teachers must likewise know how to deliver a lesson in which strategies can be discussed, their greater suitability than others defended or their relationships to previous problems drawn. The ultimate aim would be to generalise properties that may be of aid to learners when faced with similar situations in future.

**Learning PS and teachers’ knowledge**

Teaching PS necessarily defines students as solvers. Using a problem in a teaching situation involves two types of teacher knowledge: an understanding of the mathematical content required to solve the problem and the knowledge inherent in the notion of problem and its solution. When choosing a problem for classroom use teachers are influenced by their knowledge of their students and the extent to which it will be a problem for each. In particular, theoretical knowledge of PS is described either by the stages students follow in that process or the various pathways they chart to reach a solution. Such understanding enables teachers to mediate in PS by posing focused questions that provide a sound scaffolding to help students build their mathematical thinking.
By way of example, let us take a teacher who aims to teach a two-step arithmetic word problem. The problem might be as follows: “Rosa bought some sweets. She ate half and then gave five to her best friend. After that she had seven left. How many sweets did Rosa eat?” (possible calculation- or mathematical concept-related difficulties are not addressed here). In such contexts, teachers must be aware of the strategies their students may deploy to reach a solution (diagrams, dividing the problem into separate steps or other). They must also bear the difficulties in mind, which in this case may be due to the change in structure or inversion of the operation, i.e., the structural variables. Such knowledge prepares teachers to suggest alternative strategies or representations to help their students overcome such difficulties.

Teachers need to understand students’ behaviour when faced with certain problems and their possible difficulties, for such knowledge defines the limits of what can be demanded of their students. The characteristics of successful problem solvers revealed by PS research help teachers establish learning expectations (Chapman, 2015).

**Teaching PS and teachers’ knowledge**

In addition to the knowledge of mathematical tasks as problems and learning to solve them described above, teaching PS entails an understanding of how to plan and orchestrate a lesson. One element related to such understanding consists in the access pathways in PS teaching (Schroeder & Lester, 1989), commonly known as teaching about, for and via PS. Castro and Ruiz-Hidalgo (2015) contend that “the first two [for and about] deem problem solving as a learning objective and the third [via] a vehicle to teach or develop other content” (p. 95).

For instance, the aim of the mathematics teaching for PS approach is for students to acquire the ability to apply mathematical knowledge when solving problems. Teachers must therefore know how to sequence a series of tasks, initially around concepts and subsequently around transferring knowledge between contexts (Castro & Ruiz-Hidalgo, 2015). This approach requires teachers to be conversant with a type of problems that offer students different contexts in which to apply their mathematical knowledge.

In teaching about PS the goal is problem-solving process instruction, characterised by two elements. The first is related to solving models as described by Pólya (1945), which translates into a knowledge of the stages involved and their implications for the actions to be performed by students in each. Other processes such as communicating solving strategies or representing mathematical ideas acquire relevance in this context. Teachers must also be able ensure that the transition from one stage to the next takes place naturally. They must also understand PS as a dynamic, non-linear process in which solvers may turn back to an earlier stage where necessary. A second element has to do with specific strategies (such as look for a pattern or make a table). Teaching about them calls for teachers’ knowledge of the types of problems that foster the strategy...
they want their students to learn but do not mandate or constrain the freedom to choose or invent another. Teachers must also understand how affective factors may impact the use of the strategy they aim to teach. The effect of the mandatory use of a certain strategy under teacher instructions differs from that of encouragement and discussion of its use and assessment of its relevance or efficacy.

The via PS approach, in turn, is used as a teaching method and a vehicle for learning classroom mathematics (Castro & Ruíz-Hidalgo, 2015). The specific aim is for students to build classroom mathematics via problematisation. Teachers must be acquainted with the for and about approaches, but primarily be good problem selectors or designer, for the mathematics their students will be able to build will depend largely on those skills. They must envisage the representations resorted to by students to solve problems and how to guide them if they run into difficulties. This is a complex approach, for a number of factors intervene in its application, some relating to mathematics and others to processes but primarily to teachers’ beliefs about how mathematics should be learned. Non-cognitive factors are essential for structuring this approach and allowing students to express the knowledge acquired by exploring, discussing and defending their work.

Each approach requires a specific type of teachers’ knowledge, as clearly shown by the foregoing analysis of problems, their solution and learning, as well as of the approaches themselves.

**Choice of the knowledge models analysed**

To choose the models addressed in the present analysis we reviewed the literature, “it makes clear where new ground has to be broken in the field and indicates where, how and why the proposed research will break that new ground” (Cohen et al., 2018, p. 162). The review identified the teachers’ knowledge models most widely used in mathematics education research. The characteristics of the review and the extent to which they afforded a response to our research question are described below in terms of the taxonomy proposed by Randolph (2009).

The focus is a critical discussion of teachers’ knowledge for teaching PS. We position this study in the framework of traditional reviews—which usually adopts a critical approach (Jesson et al., 2011). It differs, however, in that it furnishes information on how the sources were identified, what was included, what excluded and why. That modus operandi helps identify weaknesses in or reveal the insufficiency of today’s theories or document the absence of theory, which would justify putting forward a new theory (Randolph, 2009).

As Randolph (2009) suggests with regard to qualitative reviews and following our research question, we have adopted a perspective in which we conjecture that the mathematics teachers’ knowledge models used do not address the fundamental factors for PS. That premise led us to draw a sampling of typical cases (Hernández et al., 2014), because we called for “an abundance of in-depth, high quality information rather than quantity or standardisation” (Hernández et al., 2014,
The outcome was the identification of the theoretical models for teachers’ knowledge described in mathematics education handbooks published in 1990-2018. The chapters explicitly referring to teachers’ knowledge in the title or abstract are listed in the column headed “Chapter/s” in Table 2. The 14 models resulting from the review are shown under the “Knowledge model” column in the same table.

Table 2. Knowledge models present in handbooks

<table>
<thead>
<tr>
<th>Handbook</th>
<th>Chapter/s</th>
<th>Knowledge model</th>
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<tbody>
<tr>
<td>Handbook of Research on Mathematics Teaching and Learning (Grouws, 1992)</td>
<td>Chapter 8</td>
<td>Teachers’ knowledge: Developing in context (Fennema &amp; Franke, 1992)</td>
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<tr>
<td>International Handbook of Mathematics Education (Bishop, Clements, Keitel, Kilpatrick, &amp; Laborde, 1996)</td>
<td>Chapter 29</td>
<td>None described</td>
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<tr>
<td>Handbook of International Research in Mathematics Education – 1st ed. (English, 2002)</td>
<td>Chapter 10</td>
<td>None described</td>
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<td>Second International Handbook of Mathematics Education (Bishop et al., 2003)</td>
<td>Chapter 22</td>
<td>Topology of professional knowledge (Bromme, 1994)</td>
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<td>Chapter 23</td>
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<td>Chapter 15</td>
<td>Mathematics teachers’ professional knowledge (Ponte, 1994)</td>
</tr>
<tr>
<td>Second Handbook of Research on Mathematics Teaching and Learning (Lester, 2007)</td>
<td>Chapter 4</td>
<td>MKT (Ball et al., 2008)</td>
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<tr>
<td>Handbook of International Research in Mathematics Education – 2nd ed. (English, 2008)</td>
<td>Chapter 10</td>
<td>MKT (Ball et al., 2008)</td>
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<td>Chapter 11</td>
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<tr>
<td>The International Handbook of mathematics teacher education.</td>
<td>All chapters</td>
<td>MKT (Ball et al., 2008)</td>
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<th>Handbook</th>
<th>Chapter/s</th>
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<tr>
<td><em>Volume 1: Knowledge and beliefs in mathematics teaching and teaching development</em> (Sullivan &amp; Wood, 2008)</td>
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<td>Teachers’ mathematics-for-teaching (Davis &amp; Simmt, 2006)</td>
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<td>Teacher knowledge and mathematics teaching (Chinnappan &amp; Lawson, 2005)</td>
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<td>Teachers’ knowledge: developing in context (Fennema &amp; Franke, 1992)</td>
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<td>Mathematics teachers’ pedagogical content knowledge (An et al., 2004)</td>
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<td>Pedagogical content knowledge in mathematics (Marks, 1990)</td>
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<td>The COACTIV Project (Kunter et al., 2013)</td>
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<td>KQ (Rowland et al., 2005)</td>
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<tr>
<td><em>Third International Handbook of Mathematics Education</em> (Clements et al., 2013)</td>
<td>Chapter 12</td>
<td>MKT (Ball et al., 2008)</td>
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<td>KQ (Rowland et al., 2005)</td>
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<td></td>
<td>Chapter 13</td>
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<td><em>The Second Handbook of Research on the Psychology of Mathematics Education. The Journey Continues</em> (Gutiérrez et al., 2016)</td>
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<td>MKT (Ball et al., 2008)</td>
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<td>TEDS-M framework (Tatto et al., 2008)</td>
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<td>Mathematical Discourse for Teaching (Cooper, 2014)</td>
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<td></td>
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<td>MKT concepts and processes rewrite (Foster et al., 2014)</td>
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<td>Specialised technological and mathematics pedagogical knowledge (Getenet et al., 2015)</td>
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<td>Chapter 10</td>
<td>Framework for analysing pedagogical content knowledge (Chick, et al., 2006)</td>
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<td>Handbook</td>
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<td>MKT (Ball et al., 2008)</td>
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<td></td>
<td></td>
<td>The COACTIV Project (Kunter et al., 2013)</td>
</tr>
</tbody>
</table>

The criterion for determining the most widely used models was their presence in at least two of three databases / search engine: Google Scholar (GS), Web of Science (WoS), and Scopus. We have chosen these three tools because, as Williams and Leatham (2017) point out, the indexes contained in it are often used to measure impact from a citation-based perspective. On the other hand, by drawing from more than one database, we were able to minimise the effects of issues such as: (a) citations not necessarily related to the use of models; (b) the absence in WoS and Scopus of some papers included in GS; and (c) the less demanding requirements for a GS than a WoS or Scopus listing. After identifying the key paper in which each knowledge model was proposed, we determined the number of citations received by database (see Table 3).
With the highest number of citations in all three sources, MKT was deemed the model most widely used. The next highest scores in GS were found for Teachers’ Knowledge in Context, Pedagogical Content Knowledge in Mathematics and the KQ. As the second mentioned, Pedagogical Content Knowledge in Mathematics, does not envisage all the dimensions of teachers’ knowledge, it was excluded. The first, Teachers’ Knowledge in Context, appeared in only one database, whilst KQ appeared in two of the three consulted (in second place in Scopus). The KQ was consequently determined the second most widely used teachers’ knowledge model.

In addition to drawing from the same source (Shulman, 1986), one of the two models chosen, MKT, adopts the perspective of in-service teachers, whilst the KQ analyses teachers’ pre-service training. That distinction ensured that they did not distort but rather complemented the aim of revealing the limitations of these theoretical frameworks in connection with mathematical processes such as PS.

Table 3. Teacher knowledge model citations

<table>
<thead>
<tr>
<th>Knowledge model</th>
<th>Citations in GS</th>
<th>Citations in WoS</th>
<th>Citations in Scopus</th>
</tr>
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<tr>
<td>Pedagogical Content Knowledge in Mathematics (Marks, 1990)</td>
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<td>164</td>
<td>189</td>
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<tr>
<td>Teachers’ Knowledge: Developing in context (Fennema &amp; Franke, 1992)</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Topology of teachers’ professional knowledge (Bromme, 1994)</td>
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<tr>
<td>Mathematics teachers’ professional knowledge (Ponte, 1994)</td>
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<td>Mathematics Teachers’ Pedagogical Content Knowledge (An et al., 2004)</td>
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<tr>
<td>KQ (Rowland et al., 2005)</td>
<td>536</td>
<td>155</td>
<td></td>
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<tr>
<td>Teacher Knowledge and Mathematics Teaching (Chinnappan &amp; Lawson, 2005)</td>
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<tr>
<td>Mathematics-for-Teaching (Davis &amp; Simmt, 2006)</td>
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<td>109</td>
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<tr>
<td>Mathematics for Teaching (Adler &amp; Davis, 2006)</td>
<td>287</td>
<td>89</td>
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<tr>
<td>Framework for analysing Pedagogical Content Knowledge (Chick et al., 2006)</td>
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</table>
Table 3. Teacher knowledge model citations

<table>
<thead>
<tr>
<th>Knowledge model</th>
<th>Citations in GS</th>
<th>Citations in WoS</th>
<th>Citations in Scopus</th>
</tr>
</thead>
<tbody>
<tr>
<td>MKT (Ball et al., 2008)</td>
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<td>1397</td>
<td>1275</td>
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<tr>
<td>TEDS-M Framework (Tatto et al., 2008)</td>
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<tr>
<td>The COACTIV Project (Baumert &amp; Kunter, 2013)</td>
<td>149</td>
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<td></td>
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<tr>
<td>Mathematical Discourse for Teaching (Cooper, 2014)</td>
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<td></td>
</tr>
<tr>
<td>Specialised Technological and Mathematics Pedagogical Knowledge (Getenet et al., 2015)</td>
<td>5</td>
<td></td>
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</tr>
</tbody>
</table>

*Note: blank cells mean the publication was not included in the database.*

After identifying the models to be reviewed, MKT and the KQ, we analysed the knowledge components or dimensions explicitly defined and where in each dimension PS knowledge was positioned. More specifically, we conducted a detailed analysis of the two knowledge models, testing the capacity of each to identify PS knowledge based on knowledge about problems, PS and their instruction.

**Results**

In the following sections, we discuss how PS knowledge is addressed in the two models and their respective dimensions and categories in the subsections below.

**MKT viewed from the PS perspective**

MKT (Ball & Bass, 2009; Ball et al., 2008; Hill & Ball, 2009; Thames & Ball, 2010), comprise two dimensions or domains: content knowledge and pedagogical content knowledge.

*Content knowledge:* Those authors defined content knowledge as the classroom mathematics needed to solve the problems posed to students (Ball et al., 2008). A first subdomain within this domain is common content knowledge, defined as “the mathematical knowledge and skill used in settings other than teaching” (Ball et al., 2008, p. 399), recognising that “some of the mathematical resources that teaching requires are similar to the mathematical knowledge used in settings other than classrooms” (Thames & Ball, 2010, p. 223), that enable teachers to know “whether a student’s answer is correct” (Hill & Ball, 2009, p. 70).
In addition to common content knowledge, Ball et al. (2008) identified teacher-specific or specialised content knowledge. That subdomain, acknowledged to be one of this model’s major contributions, can be equated to a way of understanding mathematics from a classroom perspective. It differs from the command of other more scientific or technical knowledge and is independent of students, instruction and curriculum (Thames & Ball, 2010). It is knowledge that views the concept from a different perspective that is useful and necessary for understanding it.

The third subdomain is knowledge of the mathematical horizon, described as “peripheral vision” (Thames & Ball, 2010, p. 224) that affords a broad view of the implications of and interconnections among the concepts taught (Hill & Ball, 2009).

In the PS context, common content knowledge as construed by the authors would consist in teachers’ own PS skills. The question that might be posed here is: which particulars of PS are common content and which specialised content knowledge? Whilst the argument is clearly logical, authors such as Carrillo et al. (2018) note the difficulties involved in differentiating the two types of knowledge. For instance, which types of PS constitute common and which specialised knowledge? Plausibly, a command of one type might be thought to be common, and awareness of the existence of all types to be specialised, knowledge. The determination of which types are deemed common and which specialised knowledge is unclear, however.

The question that might be posed around specialised knowledge is: which aspects of PS constitute teachers’ knowledge unrelated to the ability to solve a problem or to teach PS? One of the fundamental characteristics of problems, that the task to be performed must be challenging for the solver (Chapman, 2015; Lester, 2013), necessarily involves students, which clashes with the description given by the model’s authors for this subcategory.

Lastly this process evolves over time and constitutes a personal construct involving both cognitive and non-cognitive elements that cannot be unequivocally determined for certain ages. Those two factors render its analysis in terms of the mathematical horizon category particularly complex.

Pedagogical content knowledge: A second domain is pedagogical knowledge, described as a combination of content knowledge and general pedagogical knowledge (Thames & Ball, 2010). The authors note that certain “subdomains that combine knowledge of content with knowledge of students, teaching and curriculum” (p. 223) can be identified in this domain. The first subdomain, knowledge of content and students, focuses on students’ most common conceptions, errors and difficulties around given types of mathematical content. That entails not only identifying the error and its nature, but recognising it as a common difficulty and planning tasks accordingly (Ball et al., 2008). A second subdomain is knowledge of content and teaching, defined as a combination of knowing how to teach and knowing mathematics (Ball et al., 2008). The authors describe it as knowledge underlying suitable decision making in terms of examples, tasks or assessment with which a concept can be learned. The third subdomain, knowledge of content and curriculum, entails familiarity with official curricular proposals geared to student learning. Ball & Bass (2009)
explain that it is a detailed view of the school curriculum, narrower than the peripheral vision that describes knowledge of the mathematical horizon, for it focuses on standards, their interconnections and supplementary materials.

In PS, the description of content and students naturally leads to the various characterisations of solvers. Research in this area not only premises erroneous ideas, however, but focuses on good solvers. Chapman (2015) stresses the need to differentiate between knowledge of the characteristics of good solvers and knowledge of possible difficulties, bearing in mind solvers’ thought processes. The author emphasises that these elements are useful when understood from the students’ perspective and highlights the importance of making what they know meaningful as they solve problems.

Knowledge of content and teaching would include a knowledge associated with problems, PS, teaching approaches, and assessment. Such breadth renders it difficult to broach, for all these particulars would arise in classroom situations, reducing the feasibility of detailed analysis useful for understanding both the nature of such knowledge and its implementation by teachers.

A knowledge of PS in the curriculum would entail a command of the levels or grades associated with certain skills involved in PS, how they connect with other areas of the curriculum or how they are addressed in textbooks. PS, however, is a personal, non-standardisable process that develops slowly.

Although as Foster et al. (2014) contend, the MKT model can identify elements of knowledge, inasmuch as its logic stems from a mathematical concept certain elements are omitted from the perspective of PS. Given the way they are structured, these knowledge domains fail to envisage essential elements of the process, such as the solver perspective, generating difficulties in categorisation. Moreover, certain pivotal elements in PS teaching are absent when PS is discussed on the grounds of this model. Class orchestration, an imperative for authors such as Lester and Cai (2016) for instance, is excluded from this classification.

**The KQ viewed from the PS perspective**

Like MKT, the KQ builds on Shulman’s (1986) ideas. Having been spawned in a context of classroom practice under a pre-service training programme, it is evaluative in nature and the dimensions used differ from the traditional distinction between content knowledge and content teaching. As it envisages teachers’ knowledge as knowledge-in-action, its dimensions are apt for identifying knowledge-in-use (Rowland et al., 2005; Turner & Rowland, 2011).

The first dimension under this model, foundation, is related to teachers’ previous knowledge and beliefs. Turner & Rowland (2011) note that it refers to knowledge, understanding and the resources learnt in different stages of training. It differs from the other three dimensions because it is
knowledge (intentionally or unintentionally) held. The authors explain that in this dimension the three key subdivisions of knowledge and understanding are mathematics per se; significant factors resulting from research; and beliefs about mathematics, including how and why it is learned (Rowland et al., 2005).

The other three dimensions relate to knowledge-in-use. The second, transformation, describes action geared to students pursuant to judgement based on the first dimension, foundation (Rowland et al., 2005). That would translate into choosing examples, representations and proofs (Turner & Rowland, 2011). Connection, the third dimension in this model, defines the relationships between mathematical elements that are coherent with their internal logic (Turner & Rowland, 2011). It covers anticipation of complexity, decisions around sequencing, possible connections and recognition of conceptual suitability. The fourth dimension, contingency “concerns the teacher’s response to classroom events that were not anticipated in the planning” (Turner & Rowland, 2011, p. 202), in other words deviations from planning, possible replies to students’ ideas and their use as learning opportunities.

From the PS perspective, foundation-related knowledge would comprise teachers’ own PS skills, theoretical factors stemming from research on PS and their beliefs about the process. It would also include elements associated with approaches or access pathways. This single dimension consequently mixes elements relating to problems and to teaching and beliefs about both. While we feel that addressing beliefs about PS is important, this approach seems too broad for any useful analysis of the knowledge included in this dimension.

Transformation, in turn, is related to problem selection, possible ways to solve them and so on, factors that overlap with the elements included under foundation. In keeping with the authors’ description of transformation, elements associated with solver characteristics should appear, but that does not occur naturally. The third dimension, connection, refers to the relations between process and concepts. That description also prompts overlap, however, in terms for instance of the teaching approaches that implicitly favour certain sequences or types of lessons and appear in other model dimensions. Lastly, whilst contingency would identify knowledge related to orchestrating a PS lesson, elements such as discourse (Lester & Cai, 2016) are omitted.

One positive factor is that KQ identifies elements associated with PS lesson orchestrating and teachers’ beliefs, both of which are relevant to PS teaching (Lester & Cai, 2016). Nonetheless, this framework may omit some elements identified earlier, such as the characteristics of successful problem solvers.

**Discussion**

PS research, a fruitful field of study, has established useful knowledge on the role of PS in teaching and learning in mathematics classrooms. Its contributions are introduced into classrooms very
slowly, however, and at times only partially or incompletely. The results of this study reveal a steady proliferation of new frameworks, as set out in the handbooks published over the years. However, this area of research has led to the relegation of mathematical processes. Theoretical models of teachers’ knowledge have not focused on this concept. Specifically, recent and non-well known proposals have been forthcoming for explicit reflection on PS as a component of teachers’ knowledge. The proliferation of frameworks such as MKT and the KQ constitute progress that has not yet been extrapolated to PS.

This present fine-grain focus identifies the shortcomings in teachers’ knowledge of PS in the two most commonly used and studied models. As both are guided by mathematical concepts, they omit pivotal factors that underlie processes as PS. They therefore fail to envision how processes, which differ in nature from concepts, align with the components of teachers’ knowledge, and to establish proposals in which PS is explicitly analysed as part of teachers’ knowledge. Against that backdrop, the present findings support the following premises.

◆ Positioning content knowledge in a dimension outside teaching itself proves useful for analysing the knowledge of concepts, but not of PS or how it is taught. For instance, a number of studies, conducted from different perspectives, have analysed what teachers know about the concept of fraction (Castro-Rodriguez et al., 2016). That cannot be extrapolated to problems, however, for a problem acquires meaning through the relationships established with the student solving it. The present analysis reveals a limitation of the most commonly used models to focus on the student-content relationship and therefore on the imperative relationship between student and problem. Therefore, failure to consider the student can lead to an authentic teaching of problem solving not being done, and to reduce the teaching to solving exercises.

◆ In the MKT model a task is not identified as a problem for teachers based on the student-problem relationship. That may have contributed to the lack of research on teachers’ knowledge of the matter, even where a problem is deemed a task with no direct pathway for the solver to follow. As contended by Carrillo et al. (2018), conceptualising teachers’ knowledge globally as specialised is believed by the present authors to be more suitable than separating knowledge of content from knowledge of teaching content. With such a perspective teachers’ knowledge of problems could be broached in terms of their students rather than only of themselves.

◆ The primary shortcoming to the KQ model is element overlap. Some aspects of the approaches to PS teaching arise in all its dimensions and in connection with sequencing classroom tasks, for instance. Nonetheless, research has shown that teachers primarily deploy the teaching for PS approach (Pansell & Andrews, 2017). Pansell and Andrews (2017) in fact contend that teaching through PS emerges spontaneously. In that same study they point out that when the aim is for students to explicitly learn a concept through PS,
the characteristics of that approach are not maintained throughout the lesson. This is not to
defend one approach as more suitable than the other, but only to argue that teachers need
knowledge enabling them to prioritise one over the other depending on the objectives
sought in a given lesson. Element overlap in such a context obstructs a specific, clear view
of how a lesson is organised depending on the approach, each of which is governed by a
conceptualisation of mathematics, a consideration omitted in the KQ model. Hence the
need for in-depth review of the types of professional knowledge brought to bear and their
coordination in classroom scenarios.

To date part of the literature notes that teachers’ expert knowledge is generally determined by the
student-problem relationship and assumes the existence of a core or anchor category of knowledge
deemed critical to PS teaching that imbues the other categories of knowledge with meaning
(Chapman, 2016). As the present findings show, however, the theoretical frameworks in place
omit the importance of teachers’ knowledge in that relationship. That would explain why research
on the subject has paid so little attention to teachers’ knowledge of the relationship between
students and problems. Further study is required to help understand how teachers envisage that
relationship, their degree of awareness of its existence and how they use it in the classroom
practice.

Processes and concepts are conceptualised differently (NCTM, 2000) and the knowledge models
reviewed here fail to capture the elements of the former, for their logic focuses on the components
of concept knowledge. As an example, we have analysed the teacher knowledge involved in
teaching the different types of problems, the PS process and learning and teaching PS. One of the
implications of that failure is that it limits the reach of research findings. One way to overcome
that shortcoming is for the research perspective to move from teachers’ thinking to a collective
view of teachers and students, focalising on mathematical processes such as PS. More specifically,
that analysis should be conducted at two levels, i.e., macro (classroom events) and micro, with
sights trained on teacher and student. Adopting that approach would help identify when a teacher
realises that a student is struggling with a problem and when with an exercise while at the same
time overseeing the other students participating in a PS lesson. Such research calls for two inputs:
a theoretical perspective from which to focus on teachers’ knowledge of PS; and active
participation by teachers themselves. That would yield a fuller understanding of what the
classroom use of knowledge means from a teacher’s perspective.

Conclusions

Despite the present meaningful, exhaustive analysis-supported results for research on teachers’
knowledge of PS, the present study is subject to some limitations. The primary drawback has to
do with general applicability, given the number of models analysed. Whilst the analysis omits
other prominent frameworks of knowledge, the selection criteria deployed suffice to glean
essential information, although extending the analysis to other models would enhance applicability of the approach adopted. However, we are aware that this choice is influenced by the assessment criteria of the journals. A second limitation is related to the theoretical perspective that this work takes. Although we have described the PS knowledge that would be necessary for practice, more research is still needed to help us understand how knowledge about PS is actually used in classroom practice.

Teachers’ knowledge in mathematics education has unquestionably acquired prominence in the pursuit of a general understanding of teachers’ knowledge. The present findings show, for instance, that since publication of handbooks on mathematics education in the nineteen nineties, a number of models has been put forward to ascertain what sort of knowledge is needed to teach mathematics. The present study takes a magnifying glass to such models, identifying their weaknesses in connection with the ability to detect elements characteristic of a process such as PS (rather than mathematical concepts) with a view to encouraging research on PS-related issues as an object of teaching. Thus, one of the primary conclusions of this study is the identification of a need to explicitly include such issues in the dimensions or domains of teachers’ knowledge. Such envisagement must likewise address the meaning and nature of PS as a mathematical skill demanded in today’s society.

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An Experimental Study of Collaborative Instructional Strategy (CIS) for Teaching Mathematics at Primary level in Pakistan

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Abstract: Modern concept of education is based on students’ centered learning approaches where collaborative instructional strategy is an emerging approach. It has been tested in different subjects and its effectiveness has been prooved. Therefore, this experimental study investigated the effects of Collaborative Instructional Strategy (CIS) on mathematics achievement of fifth grade students. The experiment was conducted at a Government school in District Swat, Pakistan using pre-test post-test comparative group design on 64 students in two groups (control and experimental). Mathematics Attainments Test (MAT) was developed to measure students’ academic achievement. Collaborative mathematics instructional lesson plans (CMIL) were also developed to teach mathematics. The collected data were analyzed through mean, standard deviation, pair sample t test and independent sample t test. The results of the experiment showed that Collaborative Instructional Strategy (CIS) has a significant positive effect on the academic achievement of Primary school students in the subject of mathematics. It was recommended that Collaborative Instructional Strategy (CIS) may be used to teach mathematics at primary level.

Keywords: Collaborative Instructional Strategy (CIS); Teaching Mathematics; Academic Achievement; Primary Level

INTRODUCTION

Mathematics plays a significant part in promoting logical thinking, enhances critical thinking and develops problem-solving abilities in students. It describes various numbers and shapes systems (Ma, 2009). Mathematics is a sequential subject where the students learn in sequence i.e., previous learning provides a base for learning new concepts and skills (Mulligan, Mitchelmore, & Crevensten, 2013). It means that in order to learn how to multiply, one should already have learnt how to add two numbers. Similarly, in order to learn how to divide, subtraction provides foundations. Mathematics has always been seen as a discipline that sharpens the intellect including systematic, logical and precise thoughts. In the earliest years of 21st century, experts of mathematics found that it is the integral part of human life. The famous educationalist Frobel and Montessori had the view that mental and cultural development of an individual depends on his/her
Mathematics plays a predominant role in our daily lives and has become an important element in the development of our world today.

Mathematics aims at the improvement of reasoning and logical cognitive abilities that enable the individuals to solve numerical and mathematical problems. Every Businessman, Banker, Medical doctor, Laborers, Vendors requires mathematical abilities to fulfill the requirements of his everyday life. Mathematical knowledge and skills are also used in different fields such as Genetics, Physics and Chemistry that calculate various formulas (Morsanyi, McCormack, & O'Mahony, 2018).

Mathematics teaching can only be defined as truly efficient, when it positively contributes to the learning of students. It is believed that an effective teaching method is the one that positively contribute to students’ academic performances (Ma, 2009). To analyze mathematical concepts with a reasonable degree of certainty skillful mathematics teachers implemented a number of teaching techniques, approaches and tools to fulfill the criteria of various learning needs of students. Mathematics can be explained using a step-by-step approach to the subject (Akhtar, Rashid, & Hussain, 2020). It should never be taught as a collection of separated facts and formulas. Students and teachers consider mathematics to be a challenging subject, as they face many problems in the process of teaching and learning (Portman & Richardson, 1997). In general, mathematics is considered the driest discipline at the school, consisting of repetitive, demanding, dull, arcane and meaningless calculations that have little to do with exploration and creativity (Makroo, & Dahiya, 2014).

Teachers of mathematics around the world use a variety of teaching strategies to make teaching learning process more productive. Enríquez, de Oliveira, and Valencia (2018) found that teachers use a range of teaching strategies in the execution of mathematical activities. These strategies were; pre-instructional, co-instructional and post-instructional. Asuncion-Atupan (2013) pointed out various teaching strategies which were used in teaching of mathematics. These include; interactive, innovative, integrative, inquiry based, collaborative, experiential, meta-cognitive and reflective. Baig (2015) pointed out varieties of teaching methodologies that are used in teaching mathematics across the world. Some of these methodologies are inductive, deductive, lecture, problem solving, and activity based.

Furthermore, experts didn’t impose any restrictions on the use of instructional methods in teaching mathematics but it needs to be in accordance with the demands of a particular syllabus unit, available teaching resources, nature and number of students in a class. Various teaching methods, their advantages and disadvantages along with the applications of each teaching method of mathematics were employed by researchers across the globe. Banning (2005) identified numerous mathematics teaching approaches used worldwide, such as teacher-centered (deductive), learner-centered (inductive), consultative and collaborative, content-focused, and classroom-focused approaches. Sindu (2010) discussed different teaching techniques for mathematics, such as verbal
work, handwritten work, drilling, assigning homework sheet, individual self-study and group activities. Le Donne, Fraser, and Bousquet, (2016) found three underline strategy namely active learning, cognitive learning and teacher directed method. Active learning promotes engagement of students in their own learning thereby encouraging discussions, group work and cooperation. Cognitive learning promotes stimulation and critical thinking among students and making them able learners to identify their issues. This method encourages students to a great extent to exhibit their creativity in classroom. This method also ensures maximum participation of students in teaching learning process.

Team teaching is the essence of collaborative instructional strategy. In Team Teaching, the responsibility of teaching the students is distributed among two or more teachers. Both teachers carry out different tasks at a time, for example one of the teacher speaks while the other one writes on the board. At a time, two or more teachers are involved in the teaching process. They may or may not be present in the same classroom. For example, one teacher may be teaching the class, while the others may be planning. Both instructors have to deliver similar lessons to the students. For example, one teacher writes the lesson on the board while the other shows to the class related content on a chart. Team teaching includes the general instructor, as well as the special instructor planning and instructing academic subject content to all the students in the class together. Another form of team teaching is taking turns preparing for lessons to change the pace and emphasis while at the same time the other one is monitoring. One teacher will eventually lecture while the other offers examples to help in clarification key features (Knackendoffel, 2005).

According to Badiali and Titus (2010), “the term team teaching has been used to describe some sort of teacher collaboration regarding teaching.” In this case, synchronous team teaching is regarded as the closest form of instructional relationship because it holds the biggest amount of mutual liability. Team teaching provides innovation and creativity in working with another teacher. Synchronized teaming happens when both teachers are involved instructing to the whole class of students. They synchronize their teaching by showing curriculum together. Bauwens and Hourcade (1997) stated that through team teaching, the initial implementation of new material is shared by two teachers who jointly prepare and deliver the intended academic subject matter to all students as simply and briefly as possible. At different times each may take primary responsibility for particular kinds of teaching or parts of the curriculum.

On the other hand, in Pakistan, teachers of mathematics use traditional teaching strategies for teaching which are deductive or teacher centered strategies. These strategies don’t allow the teacher to link mathematics to daily life. In Pakistani classrooms teachers usually begin the topic by selecting formulas and asking students to memories those formulas. The teachers do not explain the background and concepts of the topic they are teaching (Amirali & Halai, 2010).

In contrast to deductive teaching strategies, collaboration is a successful inclusive strategy where multiple teachers share their knowledge and skills in one or more classrooms (Lin, & Xie, 2009).
Collaborative instruction is an instructional strategy in which teachers work together on specific subject. In collaborative teaching strategy teachers help each other to solve a particular problem in day to day instructions. In countries with an advanced educational system such as United States, Australia, Canada, United Kingdom teachers use the inductive teaching strategies like collaborative instructional strategy.

Researchers have proven conclusively that collaborative instructional strategy enhanced learning of the students in advanced countries (Fatimah, Rajiani, & Abbas, 2020; Kahiigi Kigozi, et al, 2012). It enhances students’ prospect towards logical arguments, offers reasoning and consistency in critical thinking in various circumstances. Teachers also acquire valuable knowledge in their subject when working collaboratively with one another. Its importance lies in the combined efforts used to achieve common instructional objectives and resolve the issues. It is also useful in alleviating and elaborating the mathematical concepts of students. Most instructors, department heads, parents, teachers and students believed that collaborative teaching strategy is successful in teaching learning processes (McDuffie, Scruggs, & Mastropieri, 2007).

Research studies also suggested that collaborative instructional strategy is more successful than solo instruction. Wadkins, Wozniak and Miller (2004) suggested that in teaching mathematics collaborative instructional strategy increases not only the bilateral confidence and reverence between teaching colleagues but also seeks to have a significant effect on the academic attainments of learners. Hence, the study intends to use Collaborative Instructional Strategy (CIS) for teaching mathematics in Pakistan to find out its effects on the mathematics achievement of primary school students. Therefore, this experimental study was designed to investigate the effects of Collaborative Instructional Strategy (CIS) for Teaching Mathematics at Primary level in Swat, Khyber Pakhtunkhwa, Pakistan.

**METHODOLOGY**

This research study was carried out for six weeks in government primary school in Barikot Swat, Pakistan. Sixty four (64) male students of 5th grade took part in the experiment. These students were aged between 10 to 12 years and they belonged to different family backgrounds. Pre-test post-test two-group comparative experimental research design was utilized to examine the effectiveness of Collaborative Instructional Strategy. The experiment was conducted by the principal author and two volunteers from the sample school. The researcher explained the objectives of the study and the details of the course content to be covered in the study to the teachers. The collaborative teacher was trained on of Collaborative Instructional Strategy. He was informed about the general methodologies and techniques of collaborative teaching and his role was explained to him in the teaching process.

Both the control and experimental groups were instructed in separate classrooms. The principal author and one math teachers instructed the experimental group using Collaborative Instructional Strategy, while a control group was instructed by one teacher. The two teachers teaching to
experimental group mutually planned and delivered the lesson plans. They implemented the collaborative mathematics instructional lesson plans (CMIL). The lesson plans were designed in such a way that both teachers presented lesson by collaborative ways that is, one teacher teach one assist. Figure 1 diagrammatically represents this concept.

Figure 1: Collaborative teaching Models, one teach one assist

Mathematics Attainments Test (MAT) was developed to measure the achievements of students. The questions in this test were selected from 5th grade mathematics text books of Khyber Pakhtunkhwa and Punjab. The test consisted of Fifty (50) items, divided into two sections to test two aspects (strands) of mathematics, i.e. Numbers and arithmetic operations (Addition, Subtraction, Multiplication, and Division) and Highest Common Factors (HCF) & Least Common Multiples (LCM). Table-1 shows specification of Mathematics Achievement Test (MAT).

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<td>No. of Test items</td>
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<td>No. of Test items</td>
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<td>20</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>13</td>
<td>8</td>
<td>9</td>
<td>30</td>
<td>43.33</td>
<td>26.67</td>
</tr>
</tbody>
</table>

Table 1: Specification Table of Mathematics Achievement Test (MAT)

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Pilot test was carried out in Government Primary School Aboha, Swat. The mathematics achievement test was administered to a group of eighty (80) students. Item difficulty index (P) and item discrimination index (DI) were used for item analysis. In the first place the initial selection was made on Discrimination index (DI). The items having difficulty index within the range of .1 and .9 were retained, and items having discrimination index below .1 and above .9 were rejected (Suruchi & Rana, 2014; Pande, Pande, Parate, Nikam, & Agrekar, 2013; Shad, Fatima, Fatima, & Chiragh, 2018; Badkur, Suryavanshi, & Abraham, 2017). The items having Difficulty index from 30% to 70% were retained (Patil & Patil, 2015; Shad, Fatima, Fatima, & Chiragh, 2018; Pande, Pande, Parate, Nikam, & Agrekar, 2013). The mathematics achievement test (MAT) composed of fifty (50) Multiple Choice Question (MCQs) items. Twenty items were rejected, seventeen on the basis of discrimination index (DI), two on the basis of difficulty index and one as well as on the basis of the experts’ opinions. The test was validated by a team of specialists (experts). Colomer (2008) stated that the majority opinion about an item is based on the responses of at least four judges out of the six judges. On the basis of students poor attendance four (two from experimental and two from the control group) students’ post-test scores were excluded from the study. Finally, the scores of 60 students were taken (30 in experimental and 30 in the control group).

RESULTS

<table>
<thead>
<tr>
<th>Groups</th>
<th>Pre-test</th>
<th>Mean (x̅)</th>
<th>Post-test</th>
<th>Mean (x̅)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>30</td>
<td>10.23</td>
<td>30</td>
<td>15.50</td>
</tr>
<tr>
<td>Experimental</td>
<td>30</td>
<td>9.90</td>
<td>30</td>
<td>21.23</td>
</tr>
</tbody>
</table>

Table 2: Mean scores of experimental and control groups on the Measure of Mathematical Achievement Test (MAT)

Table 2 shows that the mean scores of the control group in pre-test and post-test were 10.23 and 15.50, respectively. Likewise, the mean scores of the experimental group in pre-test and post-test were 9.90 and 21.23, respectively, which means that the mean scores of experimental group increased significantly after the treatment.

<table>
<thead>
<tr>
<th>Groups</th>
<th>N</th>
<th>Mean( x̅ )</th>
<th>S.D</th>
<th>t-value</th>
<th>P value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>30</td>
<td>15.50</td>
<td>5.36</td>
<td>−4.12</td>
<td>.000</td>
</tr>
<tr>
<td>Experimental group</td>
<td>30</td>
<td>21.23</td>
<td>5.43</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Difference between mean of post-test scores of control and experimental groups on Mathematical Achievement Test (MAT)
Table 3 illustrates the mean scores and S.D of control group was 15.50 and 5.36 and the mean scores and S.D of experimental group was 21.23 and 5.43 respectively. The t-value was – 4.12 and the p-value was .000 which is highly significant and concluded that the mean scores on Mathematical Achievement Test (MAT) of the experimental group, who were taught through collaborative instructional strategy, was better than the mean scores of the control group.

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Mean( x̄ )</th>
<th>S.D</th>
<th>t value</th>
<th>P value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>30</td>
<td>6.83</td>
<td>2.82</td>
<td>– 3.54</td>
<td>.001</td>
</tr>
<tr>
<td>Experimental</td>
<td>30</td>
<td>9.43</td>
<td>2.87</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Difference between mean scores of control and experimental groups in knowledge ability on Mathematical Achievement Test (MAT)

Table 4 illustrates the mean scores and S.D of control group was 6.83 and 2.82 and the mean scores and S.D of experimental group was 9.43 and 2.87 respectively. The t-value was – 3.54 and the p-value was .001 which is significant and concluded that the mean scores of knowledge ability on Mathematical Achievement Test (MAT) of the experimental group, who were taught through collaborative instructional strategy, was better than the mean scores of the control group.

<table>
<thead>
<tr>
<th>Groups</th>
<th>N</th>
<th>Mean( x̄ )</th>
<th>S.D</th>
<th>t value</th>
<th>P value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>30</td>
<td>3.83</td>
<td>1.97</td>
<td>– 4.49</td>
<td>.000</td>
</tr>
<tr>
<td>Experimental</td>
<td>30</td>
<td>5.83</td>
<td>1.44</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Difference between mean scores of control and experimental groups on comprehension ability on Mathematical Achievement Test (MAT)

Table 5 illustrates the mean scores and S.D of control group was 3.83 and 1.97 and the mean scores and S.D of experimental group was 5.83 and 1.44 respectively. The t-value was – 4.49 and the p-value was .000 which is highly significant and concluded that the mean scores of comprehension ability on Mathematical Achievement Test (MAT) of the experimental group, who were taught through collaborative instructional strategy, was better than the mean scores of the control group.

<table>
<thead>
<tr>
<th>Group</th>
<th>N</th>
<th>Mean( x̄ )</th>
<th>S.D</th>
<th>t-value</th>
<th>P value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control group</td>
<td>30</td>
<td>4.83</td>
<td>1.62</td>
<td>– 2.32</td>
<td>.024</td>
</tr>
<tr>
<td>Experimental</td>
<td>30</td>
<td>5.97</td>
<td>2.13</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6: Difference between mean scores of control and experimental groups in Application ability on Mathematical Achievement Test (MAT)
Table 6 illustrates the mean scores and S.D of control group was 4.83 and 1.62 and the mean scores and S.D of experimental group was 5.97 and 2.13 respectively. The t-value was –2.32 and the p-value was .024 which is not significant and concluded that there is no significant difference between the mean scores of control and experimental groups.

RESULTS AND DISCUSSIONS

This experimental study examined the effect of Collaborative Instructional Strategy on the academic achievement of 5th grade mathematics students. The main findings of the study were that those students who were taught mathematics through Collaborative Instructional Strategy showed better performance that those taught by traditional teaching strategies. It was also found that those students who were taught mathematics through collaborative instructional strategy showed that their knowledge and comprehension abilities improved significantly that those taught by traditional teaching strategies. However, the study found no significant difference in application ability of control and experimental groups.

This means that collaborative instructional strategy is more effective teaching method for teaching mathematics at primary level. Zhang and Cui (2018) established that collaborative learning improve students’ cognitive performance, promote social interaction and positive learning behavior. The findings of this study was supported by Ahmad et al. (2017) who found that collaborative teaching method is effective and useful in teaching Calculus I. The scores for basic knowledge and knowledge of Calculus I show statistically significant increase. The findings by Othman, (2020) that collaborative teaching was more effective in development of students and teachers professionally support the findings of the current study. Qaisar (2011) conducted his research work on the impact of Collaborative teaching on 8th grade students’ achievement in mathematics revealed that Collaborative teaching significantly effects students' conceptual learning specifically their conceptual understanding and procedural knowledge. Pires (2020) investigated use of cognitive strategies in a collaborative learning environment fond that students’ use of elaboration strategies has positively correlated to academic achievement. Van Leeuwen and Janssen (2019) results show that several aspects of teacher guidance are positively related to student collaboration. Collaborative teaching has yields positive effects on the academic achievement of students in advance countries such as the United States, United Kingdom, China, Australia, Canada, etc. (McDuffie, Scruggs, & Mastropieri, 2007). The results of the study are also consistent with the results of the studies conducted by (Murawski & Lee Swanson, 2001; Jang, 2006; Goddard, Goddard, & Tschannen-Moran, 2007).

In our classrooms mathematics teaching is transmitted without the actual purpose and application, students do not know the constructive concept of mathematics. At primary level mathematics teachers only teach text books and use white board instead of concepts of everyday life. They solve mathematics only by formulas. Therefore teaching through traditional methods of mathematics is the main case of student’s low performance. The traditional mathematics teaching strategies neither yield effective results, nor produces effective mathematics graduates. The conventional
lecture method is based on how to explain mathematics. Throughout the teaching learning process, in traditional teaching strategies, the instructor has an active role and provides knowledge to students while the students remain passive.

Collaborative learning and teaching is the process of consultation and collaboration. It enhances students’ learning and strengthens the relationship among professional colleagues. In reality, there are several educational areas that require so much cooperation and coordination. This is particularly significant in inclusive classrooms, where teachers in the classroom can usually work with specialist teachers, counselors, health professionals, teacher assistants and parents.

CONCLUSIONS

The study concluded that teaching mathematics through collaborative instructional strategy (CIS) was more effective than single teacher traditional lecture demonstration at primary level and collaborative mathematics instructional lesson plans of co-teacher showed better results and enhanced students learning.

RECOMMENDATIONS

Based on the findings of this study it was recommended that;

- The collaborative instructional strategy (CIS) may be used for teaching mathematics at various levels to improve students’ academic achievement and to encourage students to avoid rote memorization in their learning. This would not be possible without the capacity building of the school teachers in teaching methodologies through trainings and workshops or refreshing courses.
- Likewise, it was recommended that the collaborative instructional strategy (CIS) may be prioritized in pre-service teaching programs especially in pedagogy of science subjects. Government may provide more teachers to primary schools in order to encourage collaborative instructional strategy (CIS) in our classrooms.
- So far, there is a very limited research work about collaborative instructional strategy in Pakistan. Therefore, research studies are recommended to investigate the usefulness of this strategy at all levels.

References


