Editorial from Bronislaw Czarnocha

The Winter ’20 issue is devoted to different processes of assessment in our mathematics classrooms. This issue is especially important in the context of Covid-19 teaching and learning. The last two items on the other hand take us to the classical Greek and Renaissance mathematics.

MTRJ issue opens with the general discussion by Correa and Haslam addressing the possibility of holistic assessment based on mathematical proficiency. The authors propose 5 strands of mathematical proficiency: conceptual understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, and Productive Disposition. They propose Action Research as the methodological approach to the teaching experiment they plan to conduct. We expect the second part of this report after the teaching experiment has been conducted.

This general introduction is followed by two reports, one by Arnal-Palacian investigating student mathematical flexibility of student-teachers at a public university in Spain, as well as the second one by Ping Ye and Bautista Maye investigating the degree of Adult connectedness under the impact of WBAB program.

The flexibility assessment tasks is the calculation of area of polygons constructed on a grid and geoplane. The nature of student solutions provides information about the degree of student flexibility. One of the results indicates that student flexibility depends on the task. That relationship needs to be investigated deeper as flexibility is one the phenomenological characterization of creativity according to Torrance.

The second report in interested in the assessment of adult connectedness under the impact of the BWAB program that is a program whose aim is “to make cities better places by reaching young people and equipping them to be leaders capable in transformation of their communities. The results are very interesting. This is the second paper submitted by the authors (vol.10 N 1, Analyzing Student Data as a Measurement of Success for Boy With A Ball).

The fourth paper in the assessment collection come from our colleagues in Nepal, who focus their attention on the development of Digital Awareness in the context of rapid digitalization of education in Nepal, and more general in South East Asia. They present the results of the large scope survey, which however might have undergone the change due to Covid-19.

The last two items draw their inspiration from the elements of Renaissance mathematics as well as from Geometric Algebra of Euclid and Al Khwarizmi. The fascinating item by Retamoso, who points to an interesting irregularity in the classical drawing of Leonardo DaVinci presenting golden
ratios in human body. Retamoso asks whether that irregularity can be corrected and provides a positive answer to the question.

Brenner and Czarnocha, on the other hand develop two new techniques in geometric manifestation of polynomial factorization with the help of Algebra tiles.

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Mathematical Proficiency as the Basis for Assessment: A Literature Review and its Potentialities

Priscila D. Corrêa, Dayna Haslam

University of Windsor, Ontario, Canada
priscila.correa@uwindsor.ca, duggand@uwindsor.ca

Abstract: Mathematics teaching and learning goes beyond computations and procedures; it rather includes complex problem-solving and critical thinking. Kilpatrick, Swafford, and Findell (2001) identify five mathematical competencies that are present in mathematics learning: conceptual understanding, procedural fluency, adaptive reasoning, strategic competence, and productive disposition. Although these competencies are named and discussed throughout mathematics literature, there is little to be said about the assessment practices that could be used to evaluate these five competencies. This paper offers a literature review that portrays the scarce ways in which mathematical proficiency is partially being used as the basis to assess mathematics. Most of the work that has been researched shows the use of the mathematical proficiency competencies in mathematics instruction and not in mathematics assessment. Using an action research approach, this study intends to have teacher participants and researchers working collectively in a classroom-based assessment methodology, which applies and evaluates assessment practices grounded on all five components of mathematical proficiency. These practices have the potential to inform teachers’ practices towards the further development of students’ mathematical proficiency.

Introduction

“It goes without saying that ‘knowing’ mathematics, in the sense of being able to produce facts and definitions, and execute procedures on command, is not enough.” (Schoenfeld, 2007B, p.64)

Mathematics learning can bring up frustrating and anxiety-driven memories of step-by-step procedures and formulas. There is a common belief that mathematics is only about memorization and procedures. Kilpatrick, Swafford, and Findell (2001) argue that, although mathematics
Involves a lot of deductive reasoning, school mathematics does not necessarily reflect that. The authors state that, for a long time, the school system involved sophisticated knowledge, but superficially, without a concern about understanding it; “mathematics learning has often been more a matter of memorizing than of understanding” (Kilpatrick et al., 2001, p.16). It is undeniable that mathematics involves procedures and formulas. However, this is far from the essence of mathematics. Mathematics involves thinking, reasoning, analyzing and conjecturing; that is why it involves also frustration. Processes of doing mathematics are not usually straightforward and exempted from errors or misleading paths.

In 2016, The Word Economic Forum anticipated that by 2020 the 10 most desirable skills to thrive would be in this order: complex problem solving, critical thinking, creativity, people management, coordinating with others, emotional intelligence, judgement and decision making, service orientation, negotiation, and cognitive flexibility (https://www.weforum.org/agenda/2016/01/the-10-skills-you-need-to-thrive-in-the-fourth-industrial-revolution/). These skills contemplate different nuances of the doing of mathematics and they differ in nature from what was required from school graduates in the past. For example, computational skills did not make it to this recent list, as it made it back in 1970 in 2nd place, and in 1999 in 12th, in the Fortune 500 list (Boaler, 2016). As a recent report from the National Research Council put it, “To be employable in the modern economy, high school graduates need to be more than merely literate. They must be able to read challenging material, to perform sophisticated computations, and to solve problems independently” (Kilpatrick et al., 2001, p.17). Indeed, Kilpatrick et al. (2001) state that “[t]he mathematics students need to learn today is not the same mathematics that their parents and grandparents needed to learn” (p.1). Simmt (2017) complements this thought highlighting that “Canada will need people with (…) stronger mathematical reasoning, stronger computational thinking skills, and the capacity to work on hard problems” (p.129). Therefore, why narrow students’ experiences within mathematics to procedural experiences? Why focus on computational skills in 2020 if what is expected from school graduates is complex problem-solving? Mathematics classes needed and still need to adapt according to these trends.

Mathematics classes might be allowing for a broader focus, involving conceptual discussions and explorations. However, when it comes to assessment, it is not uncommon to see the broader focus of mathematics focused on procedures. When assessing students’ progress in mathematics, there is often a limited focus on the students’ solutions rather than the students’ learning and working processes (Burkhardt, 2007). As Kilpatrick and Swafford (2002) mention “[m]ost current math tests, whether standardized achievement tests or classroom quizzes, address only a fraction of math proficiency — usually just the computing strand and simple parts of the understanding and applying strands” (p.32); mathematical proficiency being defined as a cohesive blend of conceptual understanding, strategical competence, procedural fluency, adaptive reasoning, and productive disposition. When mathematics assessments overvalue the procedural aspect of doing
mathematics, it favours the perception that mathematics education is only about procedures and formulas. This is reflected, for instance, in the “back to the basics” movement, which depicts the opinion of a group of people that “think of mathematical proficiency mainly in terms of procedure skill” (Groth, 2017, p.104). This practice comes with drawbacks, and a significant one is an emphasis on the idea that students should only worry about the “how” to do mathematics, disregarding the “when”, the “what,” and the “why.” Students tend to concentrate their learning efforts on what they perceive the teacher values or expects. Even when students’ experiences in mathematics classes are not solely focused on procedures, if the assessment expectation is only about knowing how to do the math, chances are that students will focus on procedures. In accordance with that, Schoenfeld (2007B) states that

teachers feel pressured to teach to the test — and if the test focuses on skills, other aspects of mathematical proficiency tend to be given short shrift. (...) Similarly, students take tests as models of what they are to know. Thus, assessment shapes what students attend to, and what they learn (p.72).

As Burkhardt’s acronym WYTIWYG says, “What You Test Is What You Get” (Schoenfeld, 2007B, p.72), which is a relevant reason why teachers should pay close attention to mathematics assessments. If assessments promote a poor engagement with mathematics learning, teachers and students might have to deal with a superficial development of mathematical skills.

Kilpatrick et al. (2001) emphasize the fundamental need for coordination between instruction and assessment (among other aspects) to foster the development of mathematical proficiency. The authors also indicate the need for more research about mathematics proficiency, both concerning its development and its assessment. In fact, the assessment of mathematical proficiency is a potential tool for the development of mathematical proficiency, instead of just a tool to report students’ mathematical proficiency at a certain point; “[m]athematics assessments need to enable and not just gauge the development of proficiency” (Kilpatrick et al., 2001, p.13). In this sense, this research is of relevance, given that it uses the notion of mathematical proficiency proposed by Kilpatrick et al. to analyze assessment possibilities that will not only holistically assess students’ mathematical learning, but that will also inform teachers’ practices to further develop students’ mathematical proficiency. As such, this study research question is posed as: In what ways does an assessment based on mathematical proficiency result in a holistic understanding of students’ mathematical learning processes, and ultimately lead to the development of students’ mathematical proficiency?

To answer this research question, this investigation will explore a classroom-based assessment methodology that investigates students’ mathematical proficiency, providing teachers with information to plan their classes accordingly. The implementation of the assessment methodology will yield data to be analyzed, aiming to better understand assessment tools that support students’ development of mathematical proficiency. This research is in its early stages and this paper focuses
on a literature review of what has been done so far in terms of mathematical proficiency in mathematics classrooms, assessments, and instruction. The next section of this paper speaks to mathematical assessment practices in general. Then, mathematical proficiency is defined, and a discussion on mathematical assessments based on the principles of mathematical proficiency is presented. Finally, an overview of the research done to date is offered, followed by final comments and research next stages.

Assessment Practices in Mathematics

Assessment practices can have a traditional and procedural approach when it comes to mathematics. If mathematics courses assess only mathematics procedural skills, one might get the impression that a procedure-based course is enough to build on students’ mathematical knowledge, given that students will probably succeed in procedure-based assessments. However, this could be a false impression given by limited assessments. Schoenfeld (2007B) explains that “[a]spects of strategy, metacognition, and beliefs are much more subtle and difficult to assess. Yet, doing so is essential” (p.72). It is interesting to notice that students who attend skills-based mathematics courses do not present achievement results that significantly differ from the achievement results of students that attend mathematics courses with a broader approach to the curriculum (Schoenfeld, 2007B). On the other hand, the former group does not tend to present good results in problem solving and conceptual tests, while the latter group tends to present good results when tested in these same skills (Schoenfeld, 2007B). It is of relevance to investigate mathematical assessments in parallel with all the different aspects involved in the development of mathematical knowledge and proficiency.

In accordance with these thoughts, Schoenfeld’s work (2007A) highlights that mathematics education researchers have a thorough grasp of what thinking mathematically and understanding mathematics encompass, and, as a result, they tend to advocate for assessment practices that are comprehensive in terms of content and processes. In contrast, assessment in mathematics classes is usually concentrated on products instead of processes. The repetition and reproduction of procedures are not sufficient to develop the mathematical skills that are expected from the students. When students mindlessly practice mathematical procedures, they are becoming proficient at utilizing a procedure without understanding the ideas that underlie the procedure (Schoenfeld, 2007A). Schoenfeld calls that an “Illusion of Competence” (p.10) because although students may think they are competent at that specific skill, they might fail if they have to deal with a slightly different problem or procedure. Because this focus on mindless practice is a long-dated modus operandi, it is extremely assimilated in students’ experiences, teachers’ habits, and people’s common sense. Yet, if teachers from elementary to university level are asked about tasks that go in the opposite direction, requiring students to mathematically model or prove something, teachers will agree that this sort of task is successful to evaluate students’ understanding (Schoenfeld, 2007A). So why wouldn’t teachers try different methods of assessment that holistically assess students’ mathematical work? Teachers can face a lot of resistance when trying new approaches
in their mathematics classes, as it can be difficult to change these deep-rooted ways of thinking about assessment in mathematics. Schoenfeld claims that in “some cases, curricular innovators have faced the problem that without ‘proof of concept’ (evidence that a non-standard approach will guarantee high enough test scores) school districts are reluctant to let people try new ideas” (p.13).

Changing the focus of assessments requires a focus on the types of assessment tasks chosen for students to complete. Therefore, when building assessments, teachers need to be mindful of the intentions of their tasks. Ramaley (2007) indicates that, according to a National Research Council’s report on assessment, it is recommended that classroom assessments should: “(a) share a common model of student learning, (b) focus on what is most highly valued rather than what is easy to measure, (c) signal to teachers and students what is important for them to teach and learn” (p.18). This understanding emphasizes assessment not only as a tool to report on students’ state of learning, but as a resource to inform students’ processes of learning. Burkhardt (2007) unpacks task intentions by discussing what teachers should care about regarding assessment. He argues that we should strive to teach to “societal goals,” and this starts with the intention of developing “thinkers” (p.84). Burkhardt offers a mathematical model for solving a math problem and argues that we often focus on the “solve phase” instead of on the “formulate,” “evaluate” or “interpret” phases (p.86). When working with students and assessing their mathematical proficiency, teachers often focus on the final product of the students’ work rather than the complex mathematical thinking that leads to that product (the process). When considering how teachers can assess a student’s performance, Burkhardt highlights one “holistic dimension, task type” (p.89). The author offers two assessment considerations: “[m]easure what is important, not just want is easy to measure” and “[a]ssess valued aspects of mathematical proficiency, not just its separate components” (p.78 and 79). In the first principle, Burkhardt argues that mathematical proficiencies cannot be assessed through traditional testing measures, like multiple-choice tests, as these methods are often riddled with bias. The reliance on such testing measures is because of their ease and readability. In the second principle, Burkhardt argues that teachers should consider the student’s entire performance instead of isolated aspects of students’ work. Suurtamm (2018) adds to the discussion, claiming that four features will better mathematics assessment: “ongoing and embedded in instruction, uses a variety of assessment strategies, reflects meaningful mathematics, and includes students in the process” (p.475).

Mathematics assessment intentions need to be visible in all aspects of teaching (planning, instruction, and assessment). The practices that teachers engage in while teaching and assessing provide students with the grounds needed to deepen their mathematics understanding. Mathematics assessment practices must have a relationship with the tasks used to evaluate student learning (de Lange, 2007). de Lange (2007) states that there is a relationship between “‘expected performances’ (or ‘competencies’ or ‘learning goals’) and assessment items” (p.100). When teachers use problem-solving methods in mathematics, they are usually focused on the question
and the solution, instead of focusing on the competencies that are present in the problem-solving process. de Lange identifies 3 “clusters of competencies” that are used in classroom assessment: “reproduction,” “connections,” and “reflection” (p.102). In the first cluster, students complete standard equations and computations to solve for a solution. In the connections cluster, students move toward problem-solving by making connections to other topics in mathematics or the context of the problem. While in the reflection cluster, students are engaging in abstract math thinking that involves “mathematical thinking, generalization, [and] abstraction and reflection” (p.102). The current study intends to go beyond the reproduction cluster and explore the connections and the reflection clusters, allowing students and teachers to further examine teaching and learning processes. As Schoenfeld (2007A) states, “[f]rom the teacher’s perspective, assessment should help both student and teacher to understand what the student knows, and to identify areas in which the student needs improvement” (p.9).

Although assessment tasks and practices have been discussed at length (e.g. Pai, 2018; Straumberger, 2018; Swan & Foster, 2018), there is very little research that discusses mathematical assessment practices, in which teachers assess all five strands of mathematical proficiency: conceptual understanding, procedural fluency, adaptive reasoning, strategic competence, and productive disposition. For this research study, the goal is to work with elementary students and teachers to provide enriched mathematical tasks combined with a holistic mathematical assessment that evaluates the five mathematical proficiencies. Indeed, mathematics education research indicates that teachers and students can benefit from holistic assessment practices that go beyond the reproduction of procedures and is not solely product-based (e.g. Foster, Noyce, & Spiegel, 2007). Besides, de Lange (2007) argues that rich, holistic assessments should begin in the early years of education through to secondary education. Unfortunately, this is not a common practice when it comes to mathematics education. As Suh (2007) states, in “many traditional classrooms, procedural fluency plays a dominant role in defining mathematical proficiency” (p.164). As a result, instead of encountering holistic assessments, students encounter procedural assessments. This research aims to explore holistic assessments that identify students’ areas of improvement and inform teachers’ practice towards students’ development of mathematical proficiency.

Mathematical Proficiency-Based Assessments – An Option

But how does one begin to create assessments that are framed by mathematical proficiency? How mathematical proficiencies should be tested? What types of assessments should teachers be using? To conjecture about these questions, it is necessary to first understand what exactly defines mathematical proficiency.

According to the website dictionary.com (accessed on October 19, 2020), the word proficiency can be defined as “the state of being proficient”; proficient can be defined as “well-advanced or competent in any art, science, or subject”; and competent can be defined as “having suitable or sufficient skill, knowledge, experience, etc., for some purpose.” Based on these definitions, being
mathematical proficiency could be defined as *having suitable or sufficient skill, knowledge, or experience in mathematics*. Kilpatrick et al. (2001) created a framework to explain what mathematical proficiency encompasses. The authors argue that to be mathematically proficient, students need to be skilled in five different strands that contemplate concepts, procedures, strategies, reasoning and attitudes. The strands of mathematical proficiency are named as conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Each strand is briefly defined in Table 1.

<table>
<thead>
<tr>
<th>Conceptual Understanding</th>
<th>[C]omprehension of mathematical concepts, operations, and relations.</th>
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<tbody>
<tr>
<td>Procedural Fluency</td>
<td>[S]kill in carrying out procedures flexibly, accurately, efficiently, and appropriately.</td>
</tr>
<tr>
<td>Strategic Competence</td>
<td>[A]bility to formulate, represent, and solve mathematical problems.</td>
</tr>
<tr>
<td>Adaptive Reasoning</td>
<td>[C]apacity for logical thought, reflection, explanation, and justification.</td>
</tr>
<tr>
<td>Productive Disposition</td>
<td>[H]abitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.</td>
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Table 1. The strands of mathematical proficiency (Kilpatrick et al., 2001, p.116)

The five strands of mathematical proficiency, according to Kilpatrick et al. (2001), are intrinsically related to each other, to the point that one depends on the others. The correlation between these strands is organic in such ways, that it can be difficult to describe which strand(s) is(are) in place in a certain piece of students’ work. Kilpatrick and Swafford (2002) explain that

> [t]he most important feature of mathematical proficiency is that these five strands are interwoven and interdependent. Other views of mathematics learning have tended to emphasize only one aspect of proficiency, with the expectation that other aspects will develop as a consequence. For example, some people who have emphasized the need for students to master computations have assumed that understanding would follow. Others, focusing on students’ understanding of concepts, have assumed that skill would develop naturally. By using these five strands, we have attempted to give a more rounded portrayal of successful mathematics learning (p.9).

This interdependency between the strands of mathematical proficiency implies that teachers need to consider all five strands in the teaching and learning processes of their students, which includes assessments. The way students are taught, learn, and develop proficiency, as well as the way
teachers are educated to promote students’ mathematical proficiency need to mirror this inherent relation between the strands (Kilpatrick et al., 2001). The interrelation between the five strands can also give some insight into the learning difficulties students might have with mathematics in the long run. Given that mathematics proficiency encompasses five different strands and school mathematics frequently focuses on only one of them, namely procedural fluency, it would make sense that students encounter challenges or do not properly learn mathematics due to the absence of the other four strands. Kilpatrick et al. mention that, in the United States, there is a tendency “to concentrate on one strand of proficiency to the exclusion of the rest” (p.11); this tendency is not an exclusive behaviour of the United States mathematics classrooms.

Aiming to authentically assess students’ entire mathematical performance and knowing that Kilpatrick et al.’s (2001) model of mathematical proficiency considers five comprehensive nuances of students’ work, Kilpatrick et al. framework was the chosen one to support this research approaches on mathematical assessment. Kilpatrick and Swafford (2002) confirm that if students are to become mathematically proficient, different aspects of their education need to undergo significant changes, assessments being one of them.

Does improving students’ math proficiency require new types of tests?

Yes. New tests may be needed, and old tests may need to be changed. (...) Teachers need tests and other assessment procedures that let them gauge how far students have come along in all five proficiency strands. Furthermore, instead of taking time away from learning, these instruments should allow students simultaneously to build and exhibit their proficiency. (Kilpatrick and Swafford, 2002, p.32)

Because of the challenges in changing assessment practices and in accepting new assessment possibilities, this research intends to portray how assessment approaches that consider the five strands of mathematical proficiency can be beneficial in mathematics elementary classes to effectively inform teachers’ practice and foster students’ development of mathematical proficiency. In other words, the goal is to analyze assessment options that coherently and comprehensively examine students’ strands of mathematical proficiency to inform teachers’ practice, allowing for the further development or improvement of these strands. Kilpatrick and Swafford (2002) acknowledge that some changes might have already been reflected in instructional materials and even in assessments, nevertheless, they state that “progress has been uneven and poorly documented” (p.4), reinforcing the importance of research studies in this realm. In addition, Kilpatrick et al. (2001) state that they “find real progress toward mathematical proficiency to be woefully inadequate” (p.11). This research intends to contribute to the mathematics education research field by offering practical guidance and options that can contribute to the progress toward mathematical proficiency.
Mathematical Proficiency-Based Assessment and Instruction – An Overview

To the extent of this study, research surrounding the five strands of mathematics proficiency in the mathematics classroom is limited. Much of the relevant research for this study only partially considers the five strands. Besides, most of the research found relates to instructional practices. This section discusses the ways in which the five strands have been used and the findings of these studies as it relates to assessment and instruction.

Khairani and Nordin (2011) created a mathematical proficiency test to assess the development and relationship between conceptual understanding, procedural fluency, and strategic competence of 588 14-year-old students. These three strands were chosen apart from the remaining two (productive disposition and adaptive reasoning) as Khairani and Nordin argue that the excluded strands are not yet mature enough to assess and that conceptual understanding, procedural fluency, and strategic competence are easier to assess using a “standardized achievement test” (p.35). The test was structured in the following manner: 50% of the test focused on conceptual understanding, 32% of the test focused on procedural fluency, and 18% focused on strategic competence. The topics covered on the test were “Linear Equation, Algebraic Expressions II, Ratios, Rates, and Proportions I, and Coordinates and Circles I” (Khairani & Nordin, 2011, p.37), and there was no specific training or special classes to prepare students for the test. The questions on the instrument concerning conceptual understanding were deemed the easiest for students to complete. Next was strategic competence, and then procedural fluency was considered the most difficult strand for the students. Students did well with questions that stated the information “explicitly” but struggled when they had to “use their prior knowledge” to solve for solutions (Khairani & Nordin, 2011, p.41). Although this is one of the very few studies that claims to assess mathematical proficiency (based on this research scope), one could question the employed assessment tool. This research study used a set of 50 multiple choice questions despite the often-cited notion that multiple-choice questions are not an effective assessment model (Burkhardt, 2007; Boaler, 2016). Multiple-choice exams can provide an inaccurate, and superficial assessment of students’ mathematical knowledge.

Suh (2007) reinforces the idea that there is still a prevailing focus on procedural fluency in mathematics classrooms. Willing to change this scenario, she proposed five different classroom activities to elementary students focused on building on the five strands of mathematical proficiency. The first activity is called “Modeling Math Meaningfully” and intends to have students working with different representations when solving problems or doing exercises, as a way of working on conceptual knowledge. The second activity is called “Math Curse” and pretends that students are under a curse that makes them see mathematics in everything. Students are then supposed to find mathematics in their daily lives, share and discuss with their classmates. The third activity is called “Math Happenings” and consists of the exploration of likely-to-happen problems proposed by the class teacher. Finally, the last two activities are called “Convince Me” and “Poster Proofs”; these activities are meant to work on deductive reasoning, argumentation, justification, and collaboration. Suh indicates that because of the nature of the activities she was able to
differentiate her instruction according to students’ needs. She also observed a considerable change in terms of students’ productive disposition.

Groth (2017) argues that the qualitative data that can be gathered during mathematics classes is relevant for improving instruction; his research investigates classroom data through a mathematical proficiency lens. Groth’s research presents an iterative cycle: 1) qualitative data is gathered from students’ interactions; 2) this data is used to portray students’ mathematical proficiency; 3) lesson plan is designed to develop students’ mathematical proficiency; and 4) feedback about the lesson is obtained from peers and mentor. Pieces of evidence informed instructional strategies to build on students’ mathematical proficiency, and results were observed. Groth underlines situations, in which correct responses were not matching correct explanations, and vice-versa, as examples of when the final answer may not be sufficient to portray students’ mathematical proficiency. The author indicates productive disposition as the most challenging strand to observe and suggests that teaching approaches might have an important role when fostering productive disposition. Groth also mentions the challenges involved in creating tasks that promote mathematical proficiency. One of the author’s final thoughts about mathematical proficiency is that teachers “can develop the fundamental habit of mind of basing daily instructional decisions on observations of their students’ strengths and needs along the five strands of mathematical proficiency” (Groth, 2017, p.107). Groth’s study gives insight into the current study in the sense that it suggests that the strands of mathematical proficiency should permeate teaching practices in the classroom. The current study intends to extend Groth’s proposition by offering an analysis of mathematics assessment practices, instead of mathematics instructional practices.

In a research study that discusses problem-based teaching models, Ozdemir and Pape (2012) describe the instructional approaches that supported students’ development of strategic competence. The teacher and the respective class were chosen for this study because the teacher had attended a Self-Regulated Learning (SRL) workshop. There is a connection between SRL and strategic competence in that students are meant to reflect on the strategies to use when given a problem and use their autonomy to carry out their plan. In their observations of the lessons and students’ learning, Ozdemir and Pape identified four categories that support the development of students’ strategic competence: “(a) the nature of tasks and activities, (b) practices supporting understanding, (c) practices supporting strategic knowledge and skills, and (d) practices supporting motivation” (p.160). Regarding the first category, the tasks that the students engaged in were intentionally planned to support student understanding through collaborative work in small groups during lessons that introduced topics. In order to support student understanding (second category), the teacher gave students detailed explanations as well as provided multiple methods of showing understanding. The concepts always had a real-life connection for students. For the third category, the teacher would invite the students to engage in group problem solving where “students exercised strategic competence by analyzing the task (e.g., rereading, under-lining, and using context clues),
selecting, adapting, and implementing strategies, as well as comparing and contrasting each other’s strategies” (Ozdemir & Pape, 2012, p.161). Finally, to support students’ motivation (fourth category) to persist in problem-solving, the teacher would acknowledge students’ understanding and highlight their strategies and ideas used to solve the problems.

Freund (2011) sought to understand teachers’ approaches when teaching for mathematical proficiency in an urban school context. The author discussed how teachers engaged in problem-based teaching to develop mathematical proficiency. The teachers in the study were filmed teaching lessons about algebraic thinking after attending a professional development seminar. Productive disposition was excluded from this study as it was deemed too difficult to measure. Based on the lessons taught and on whole class and small group discussions, the researcher observed each student’s mathematical proficiency on a scale from 1-5 to determine if the student was “proficient strong, proficient-limited, non-proficient-strong, non-proficient-limited, and no-rating-none” (Freund, 2011, p.49). According to students’ performances, the teachers were organized into three groupings: majority proficient, mixed proficiency group, and low proficiency group. In the majority proficient group, the lesson structure was formatted as follows: “problem introduction, small group work, and whole class discussion” (Freund, 2011, p.70). The teachers focused their discussions on explanations, multiple solutions, and student-to-student conversations. The mixed proficiency group shared some similarities with the majority proficiency group, while the low proficiency group had some more traditional instructional approaches, for instance, independent work and teacher-led problem-solving (Freund, 2011). It is clear to conclude that the open-ended, student-led, problem-based mathematics tasks serve students much more positively regarding mathematical proficiency development as opposed to the traditional teacher-centered model.

Student’s self-assessment of their productive disposition has been evaluated by Graven (2012). The author sought to understand how the use of mathematics clubs could influence the productive disposition of students. The students were to reflect on productive dispositions and rate themselves on a self-assessment. In this self-assessment, the students were asked to rate their proficiency in mathematics based on a 9-point scale. The students were asked to relate their performance to a hypothetical low performing student named Mpho (at the low-end of the scale), and a hypothetical high student named Sam (at the high-end of the scale). They were asked to reflect on both types of learners in terms of classroom behaviour and overall performance. The students identified that the proficiency low student, Mpho, was distracting and off task, while the proficiency strong student, Sam, was focused and completed the homework without any struggle. In the interviews, Graven found that when students were struggling with a question “only two learners suggested asking the teacher and the remaining eight learners referred to ways of solving that did not involve the teacher, e.g. ‘I thought it in my mind’, ‘I work it out’, ‘I take scrap paper or counters or my brain’, ‘I stretch my brain a bit and don’t copy’” (p.57). The students shared strategies and
struggles that they encountered in their mathematics work. These comments indicated students’ high productive disposition.

In a different perspective, Siegfried (2012) conducted a two-part study that examined productive dispositions as well, but of pre-service and in-service teachers while assessing student work. In the first study, Siegfried designed a rubric that evaluated the productive disposition of teachers that were assessing a student’s math task. In the second study, Siegfried took 10 teachers that were found to have strong productive dispositions to engage in interviews. In the first study, pre-service teachers had lower productive dispositions than in-service teachers, and Siegfried credited this to a difference in experience. In the second study, the participants that were deemed to have strong productive dispositions exhibited the following characteristics while engaging in the math tasks: determination, unstoppable effort, multi-modal approach, honesty, passionate about the excitement of mathematics, and the belief that success could be achieved by anyone (Siegfried, 2012). These qualities are arguably quite important when teaching and assessing a proficiency-based math program. These characteristics of the teacher participants in Siegfried’s study mirror students’ anecdotes in Graven’s (2012) work.

Although research does not directly address the five strands of mathematical proficiency in terms of assessment, there has been some evidence that highlights important teaching and assessing strategies that address the holistic development of mathematical knowledge. Foster, Noyce, and Spiegel (2007) discuss the ways that the Mathematics Assessment Collaborative (MAC) – a program that replaces standardized tests with a “coordinated program of support and learning for teachers based on a common set of assessments given to students” (p.138) – impacted student success. This assessment considered five main ideas about mathematics per grade level, and, in place of tests, students had five tasks to complete. These tasks “require students to evaluate, optimize, design, plan, model, transform, generalize, justify, interpret, represent, estimate, and calculate their solutions” (Foster et al., 2007, p.139) through open, problem-based tasks. This exam was given to students in grades three, five, seven, and in algebra classes in 24 school districts. The teachers evaluated each exam after receiving training on the assessment rubric. The training involved in the assessment of the tasks was a professional development exercise as teachers were taught to consider all aspects of the math task that they were assessing, and then they were expected to find evidence of that in the students’ work. The students’ exams received a grade in 4 levels, where Level 1 showed limited success and Level 4 showed success at an increased level. The exams were returned to the schools for teachers to reflect on and use in future teaching scenarios. The test was also used as “valuable information for professional development, district policy, and instruction” (Foster, Noyce, & Spiegel, 2007, p.141). For instance, if a school has shown an unsuccessful trend in a strand of mathematics, the teachers would receive professional development sessions targeting the core ideas of the strand, and later, the students would be reassessed to evaluate any changes. This was the case with a proportional reasoning task for grade 7 students. Initially, only 37% of the grade seven students were able to “meet the standard” (p.142),
and 20% could complete all the questions. This meant that 63% of the students were unsuccessful. Four years later, the students were given another proportional reasoning task and their success had increased to 59%. The exam results of the students that participated in the MAC program were compared to students that were not in the program, and even though the students in the MAC program had a lower socioeconomic status, they still were able to meet the standards on the state exam compared to those not in the MAC program. The findings support Foster et al. (2007) argument that “when teachers teach to the big ideas (…) participate in ongoing content-based professional development, and use specific assessment information to inform instruction, their students will learn and achieve more” (p.152). Furthermore, teachers were empowered by the results of these assessments as teachers were able to collaborate with other teachers to discuss strategies and student progress with other knowledgeable teachers. The MAC shows teachers where their students share similar struggles, how the big ideas are interrelated across grade levels, the importance of collaboration with other grade levels, and how to reflect on their intentions related to teaching and assessment (Foster et al., 2007).

As illustrated in the above literature review, research around mathematical proficiency and assessment is scarce. Most of the literature review refers to mathematics proficiency and instruction, and even within instructional approaches, research is still limited. Indeed, mathematics assessment practices have proven to be more challenging to change than mathematics instructional practices. Given that fostering students’ mathematical proficiency involves both instruction and assessment, this research focuses on offering new possibilities in terms of mathematics proficiency and assessment.

Researching Mathematical Proficiency-based Assessments – A Proposal

“Not only is it the case that one assessment size does not fit all, it may well be the case that one assessment size does not fit anyone’s needs very well.” (Schoenfeld, 2007A, p.14)

Challenging teaching and assessment practices can be demanding given teachers’ previous experiences as learners, beliefs, teacher education, and approaches to teaching. Besides, it is not uncommon to have teachers discouraged about new tendencies and understandings from mathematics education research, as teachers may perceive research as disconnected to teachers’ teaching practices and classroom realities. Considering this perspective and aiming to develop a study that has the potential to enlighten teachers’ approaches when it comes to mathematics assessment and proficiency, this study is planned to be an action research study that has in-service teachers fully engaged as co-researchers. This way, teachers can collaborate with researchers as they study their “own instructional methods, their own students, and their own assessments” (Mertler, 2009, p.4). As Greenwood and Levin (2007) state, action research “democratizes the relationship between the professional researcher and the local interested parties” (p.4). In the specific case of this study, the local interested parties refer to teachers concerned about assessment
practices that have the potential to assist in evaluating their students’ mathematical proficiency and inform their teaching practice in meaningful ways. The goal is to develop a practice-based research methodology with a practical purpose (McAteer, 2013), which aims to apply the five strands of mathematical proficiency to assessments more seamlessly.

This research acknowledges the relevance of integrated work between teachers and researchers. Greenwood and Levin (2007) argue that when theorists and actors (teachers in this case) are not engaged in the same way in a research study, the disengagement might do a disservice in terms of meaningful, practical, methodological and theoretical results. Action research moves towards “closing the gap between the roles of theorist and practitioner” (Kemmis, 2009, p.468, original italics). When teachers take part also as researchers, practice and theory are investigated in parallel, yielding beneficial outcomes on both ends. This desired engagement between theorists and practitioners is in tune with McNiff and Whitehead’s (2002) view of action research, in what they argue that in action research “[n]o distinction is made between who is a researcher and who is a practitioner. Practitioners are potential researchers, and researchers are practitioners” (p.15). As such, the term researchers, when used in this study, will refer to both researchers and practitioners.

Kemmis (2009) states that action research aims at transforming three things: practitioners’ practices, their understandings of their practices, and the conditions in which they practice. In this study, researchers will reflect on classroom current assessment practices, how these practices inform the development of students’ mathematical proficiency, and what are the circumstances in place, so that they can transform their practices. In other words, researchers investigate and make judgements about their own practices, in order to enhance them (Kemmis & McTaggart, 2000; McAteer, 2013). To figure out ways of improving assessment approaches, researchers can ask themselves: What strands of mathematical proficiency are or are not being considered in the current assessment practices? What changes or additions need to be made in assessment tools to ensure that the strands of mathematical proficiency are being evaluated in its entirety? How can these assessment tools help in promoting students’ mathematical proficiency? According to the outcomes of this reflection process, researchers plan an action and implement it in class. The action taken needs to be documented and analyzed, so that the plan can be modified, improved, put into practice again, and reanalyzed until researchers’ goals are achieved. This action research process has huge importance in the accomplishment of the research goals. As Greenwood and Levin (2007) argue, the “direction of an AR [action research] project is guided by the learning gained through the process, not by a set of a priori norms or expectations imposed on the situation and actors” (p.134).

McNiff and Whitehead (2002) explain that in an action research project:

[w]e review our current practice, identify an aspect we want to improve, imagine a way forward, try it out, and take stock of what happens. We modify our plan in the
light of what we have found and continue with the ‘action’, evaluate the modified action, and so on until we are satisfied with that aspect of our work (p.71).

This research study adheres to this format and follows the cycle depicted in Figure 1. The action research cycle has six distinct phases: Review, Identify, Plan, Act, Reflect, and Discuss. The initial Review stage encompasses a critical analysis of the assessment practices being used to date in the researched classes. Then, the Identify stage investigates these current assessment practices aiming to find aspects for improvement. In the Plan stage, researchers are engaged in rich planning as they modify or create teaching contexts and tools for the assessment of mathematical proficiency. This occurs both independently and in focus group discussions. The researchers then use the assessment model in the Act phase. The researchers meet again for reflection and discussion, offering adjustments to the assessment tool based on the practical applications done in the classroom. After the Reflection and Discussion phases, researchers may go back to the Plan stage to modify the initial plan as needed and reapply it. The cycle is repeated until the goals of the research are achieved. Researchers have a collaborative relationship in this project as they are actively involved in the creation and revisions of the assessment model used to evaluate the five strands of mathematical proficiency.

Figure 1. Action Research Cycle.
Final Remarks

The main goal of school mathematics education could be described as the development of mathematical skills and knowledge that will promote mathematical literacy, allowing citizens to effectively and consciously participate in society and in all that it entails. Kilpatrick et al. (2001) argue for a teaching approach that is holistic and balanced in both teaching and assessing so that “[w]hen today’s students become adults, they [can] face new demands for mathematical proficiency that school mathematics should attempt to anticipate.” (p.1). School mathematics education is supposed to form students that have a practical, but also an analytical understanding of mathematics. This paper briefly explains what mathematical proficiency is about and argues that mathematics teachers and educators should have a close look into students’ mathematical proficiency if they intend to develop long-lasting essential mathematical skills and knowledge. This research study claims that assessments focused on mathematical proficiency can be a strong ally in the development of these skills and knowledge. The understanding of students’ learning processes, through assessments that investigate mathematical proficiency, may effectively inform teachers’ practice towards the holistic development of mathematical proficiency. Grounded on a literature review about what has been done to date in terms of instruction, assessment and mathematical proficiency, this study proposes an action research classroom-based investigation that aims to fill in gaps related to the potential benefits of mathematical proficiency-based assessments.

References


Mathematical Flexibility of Degree of Primary Education students in solving an area problem: Pick’s Theorem

Mónica Arnal-Palacián

marnalp@unizar.es
University of Zaragoza, Spain

Abstract: The development in mathematical flexibility should be included in the mathematics teaching training of students in Primary Education Degree. Teachers in training have to acquire the skill to modify the problem resolution and be able to break with stereotyped methods.

This document presents an analysis of spontaneous mathematical flexibility developed by teachers in training against problems in which the calculation of the area is requested. At the same time what type of statement can promote is analysed, in a more effective way, a flexible thought and the comparison of the possible mathematical flexibility between the different problems is established.

Keywords: Mathematical flexibility; area; pre-service teaching learning; Primary Education; Pick’s Theorem

INTRODUCTION

Many students propose a single way of solving, replicating the same procedure over and over again, in the different mathematical problems posed. Possibly this is due to the absence of flexible mathematical reasoning during their school years (Joglar-Prieto, Abánades & Star, 2018). This happens at different educational levels and in different branches of knowledge of mathematics. On the other hand, the use of manipulative materials can encourage the consideration of works or strategies that had not been thought of before, awakening curiosity and reflection in teachers in training about different mathematical concepts that they will teach in the future (García-Lázaro, Garrido-Abia & Marcos-Calvo, 2020).

In particular, this study analyses the answers of 63 students of a Mathematics subject and its Didactics of the Primary Education Degree of a Spanish public university to a problem of...
calculation of the area of a simple polygon on an orthometric geoplane or grid, in both cases made in paper and pencil.

To do this, the different types of geoplane had to be introduced beforehand: isometric, ortometric and circular grid. In view of the impossibility of having enough geoplanes, and due to its dynamism, it was decided to use the free digital orthometric geoplane from Math Learning Center.

Later, the students were proposed to calculate the areas of different concave polygons, always on square frames. In the classroom, from the different activities of area calculation on a square grid, as already collected by Jiménez-Gestal and Blanco (2017), procedures of decomposition, complementation and the use of the formula of the area of the triangle emerged. Immediately afterwards, the Pick’s Theorem was introduced as an additional strategy, in which the application of a formula is sufficient.

This is intended to achieve the following objectives:

- To analyze the spontaneous mathematical flexibility of the teaching staff in training.
- To analyze what kind of statements promote flexible thinking.
- To compare mathematical flexibility in the face of different area calculation problems.

In order to achieve them, a bibliographic review of research on learning of the area magnitude and the theoretical framework that supports mathematical flexibility is carried out. Through a written and individual evaluation, two problems of calculation of areas of a simple polygon are proposed, to later carry out the analysis of the results.

THEORETICAL FRAMEWORK

This section is composed by two subsections: the first one is dedicated to the bibliographical review of the researches that analyze the area magnitude; and the second one to the mathematical flexibility, the fundamental pillar of the present research.

Learning the area magnitude

The magnitudes have a practical application in the resolution of problems, daily life situations, commercial exchanges, needs of the technical trades, that means, needs of quantification present throughout history in the different civilizations and programs of obligatory education (Chamorro, 2001). Although there is no unanimity in the different curricula, the following have always been included: usual and legal units of measurement, and knowledge and handling of measuring instruments.

Determining the area of a given figure is interesting because it involves the coordination of two dimensions. This coordination makes it possible to provide various examples: relationship with the concept of a unit and its iteration, the number of units and calculation with formulas (Outhred & Mitchelmore, 2000). In the teaching-learning processes, a qualitative and quantitative treatment
of the area must be carried out. These processes are sometimes reduced to poor instruction, reducing their determination to the use of formulas (Freudenthal, 1983).

Caviedes-Barrera, Gamboa-Rojas and Badillo-Rojas (2019), when analysing the procedures or justification, carried out by teachers in training, showed a tendency to associate the area with the use of routine calculations and formulas, even if they had to spend more time on solving the task. For this reason, these authors state that teachers in training in Primary Education have a little knowledge of some mathematical elements: unit of measurement, conservation of area, additive and multiplicative relationships. Moreover, the numerical context was easier for them than the intuitive geometric context.

This same trend is followed by Codina, Romero and Abellán (2017), who state that the importance given to the use of formulas, even at very early levels, to the detriment of understanding, hinders the development of measurement in students. This statement is supported by previous studies, such as that of Segovia, Castro and Flores (1996), who pointed out that reducing the calculation of areas in teaching to the use of formulas may justify the lack of significance of students in surface units.

The fact is that learning the area magnitude is not only memorizing formulas, but it is a complex process that requires a series of concepts, processes and skills, such as: perception, comparison, measurement and estimation (Zapata & Cano, 2008).

Mathematical flexibility

Flexibility has been studied from both psychology and mathematical education (Callejó & Zapatera, 2014). These authors, compile the different meanings of the term: as the amount of variations that a person can introduce in the notions and mental operations (Demetriou, 2004); as the ability of a person to modify the resolution of a problem by modifying the task (Krems, 1995); and as the ability to solve problems by breaking with stereotypical methods of resolution (Krutetskii, 1976). In our study we will address the second of these meanings.

Going a step further, Joglar, Abanades and Star (2017) opt for the meaning of mathematical flexibility as the capacity to produce different strategies to solve a problem and to distinguish between them the most effective for each case. However, Star and Rittle-Johnson (2008) determined that knowledge of different strategies, including the most effective one, does not imply that students are able to choose the most appropriate one for each problem posed. Therefore, a distinction is made between competence -knowledge of different strategies- and performance -choosing the most appropriate strategy for a specific circumstance.

In addition, it is important to note that flexibility can occur spontaneously or induced. The first of these refers to performance: the individual provides an innovative strategy in the first of his responses; while the second refers to competence: the person offers an innovative solution when asked to provide other solutions. Non-flexible students are those who do not provide any innovative response (Xu et al., 2017).

Among the researches that analyze mathematical flexibility, we highlight those that have teachers in training, such as the one carried out by Lee (2017) for fraction division problems. In addition,
Aguilar and Telese (2018) highlight that trainee teachers who solve a problem in different ways can better help their students. The mathematical flexibility must be made latent in the teaching staff as well. This occurs when in certain situations teachers change their plan according to the unexpected responses of their students (Leikin & Dinur, 2007).

**METHODOLOGY**

In this study we have collected, in a written form, the performance of two mathematical activities in which the area of a quadrilateral is involved.

During this section, the sample of participants, the context in which the students were, and the protocol of action followed in the theoretical classes and in the evaluation session are collected.

**Participants**

The sample has been elaborated with 63 students of a subject Mathematics and its Didactics of the Degree of Primary Education of a Spanish public university.

**Design**

In this subject, students are expected to acquire the following knowledge: measurement of a magnitude and reality; origin and historical evolution of measurement; concepts and procedures related to magnitudes and their measurement; stages in the measurement process; study of some magnitudes; and basic considerations in teaching and learning about measurement. In addition, these students enroll in the subject after passing the previous two courses in this educational branch associated with arithmetic and geometry. In particular, they must acquire the mathematical and didactic knowledge associated with the area magnitude. Among this knowledge is the decomposition into simpler geometric forms and the Pick’s Theorem.

As a reminder, Pick’s Theorem determines the area of a simple polygon, that is, its sides do not cut into each other, and its vertices are nodes in a square-weave geo-plane. The area is determined by the formula $i + \frac{b}{2} - 1$ where $i$ is the number of interior points and $b$ is the number of points on one side of the polygon.

For the latter, and in view of the impossibility of having enough geo-planes in the classroom, it was decided to use the free digital application of Math Learning Center, giving it great dynamism. It allows working with orthometric geo-planes, that is, with a square grid.

---

1 Students use $d + \frac{f}{2} - 1$ express themselves in Spanish language.
Instrument

The way to collect the development given by the students to the proposed mathematical activities was through a written test. There were two quadrilaterals from which their area had to be determined. At no time was it requested which procedure should be used to resolve it.

In the first of the exercises, a quadrilateral located on an orthometric geoplane was contemplated, and the area inside was requested to that geometric form. See Figure 1.

![Figure 1. Area activity of a quadrilateral on an orthometric geoplane.](image)

In the second one, the calculation of the inner area of a rectangle on a grid was proposed, where the latter was not visible inside the geometrical shape (see Figure 2). This task had already been proposed during the Spring Contest for 5th-6th grade Primary Education students before.

![Figure 2. Area activity of a rectangle on a grid.](image)

In addition, it should be noted that no calculation mistakes are contemplated in the resolution of the different exercises by the teachers in training, since they had a calculator in that test.
Procedure

Qualitative techniques were used in order to establish the categories in the resolution of the two tasks described, from the procedure used by the student, and thus be able to determine the spontaneous flexibility with which the teachers-in-training act. Subsequently, quantitative techniques were used to be able to analyze the frequency of each of these categories, with percentages.

RESULTS

In the presentation of the results obtained, it has been considered necessary to establish the presentation of each of the tasks separately in order to subsequently address their similarities and differences.

Results of the geoplanning activity

In the task in which it is asked to calculate the area of a quadrilateral on a square-frame geoplane, a total of two different resolutions are observed: application of the Pick’s Theorem and decomposition into right triangles of the complementary figure. In addition to these two resolution categories, it is necessary to point out the existence of teachers in training who are not capable of solving this exercise. One of the possible resolutions contemplated, because it was developed assiduously in the classroom (decomposition into simpler geometric shapes, such as right triangles and squares), was not used by any of the students.

For the first case, in which the exercise was solved using the Pick’s Theorem (see Figure 3), the students count the number of interior points, denoted as $i$ (inside) or $b$ (border); and the number of border points, those points through which the segments pass. They then apply the formula $i + \frac{b}{2} - 1$, giving rise to the inner area of the geometric form.

![Figure 3. Student Resolution. Use of Pick’s Theorem](http://www.hostos.cuny.edu/mtrj/)
In the second case, the teachers in training calculated the area of the entire geoplane and then determined the area of the four triangles outside the quadrilateral (see Figure 4). Since the latter are complementary, it was sufficient to subtract both amounts.

![Figure 4. Student Resolution. Calculation of the area with complementary triangles.](image)

\[
\begin{align*}
A_{\text{space}} &= b \cdot h = 5 \cdot 4 = 20 \\
A_{A1} &= \frac{b \cdot h}{2} = \frac{2 \cdot 2}{2} = 2 \\
A_{A2} &= \frac{b \cdot h}{2} = \frac{2 \cdot 3}{2} = 3 \\
A_{A3} &= \frac{3 \cdot 1}{2} = 1.5 \\
A_{A4} &= \frac{3 \cdot 2}{2} = 3 \\
A_{\text{quadrilateral}} &= A_{\text{space}} - (A_{A1} + A_{A2} + A_{A3} + A_{A4}) = 20 - 9.5 = 10.5
\end{align*}
\]

Among the students who failed to solve the problem we find two situations: those who did not find did not even try to find any solution to the problem and those who made a mistake. Among the mistakes we find the confusion between magnitudes, Figure 5, where the hypotenuses of triangles are calculated from the Pythagoras’ Theorem to determine the area.

![Figure 5. Wrong student resolution.](image)

\[
\begin{align*}
a^2 &= 3^2 + 2^2 = 9 + 4 = 13 \rightarrow a = \sqrt{13} \\
b^2 &= 2^2 + 3^2 = 4 + 9 = 13 \rightarrow b = \sqrt{13} \\
c^2 &= 3^2 + 1^2 = 9 + 1 = 10 \rightarrow c = \sqrt{10} \\
l &= 2
\end{align*}
\]

The frequency of these resolutions is not uniform. 41.27% of students solve the exercise using Pick’s Theorem, 7.94% with the area of complementary triangles, and 50.79% of teachers in training do not successfully solve this task.

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Among all the answers, this exercise considers spontaneous mathematical flexibility to the calculation of the area through complementary triangles, because the student must perform a reflection task. However, the application of Pick’s Theorem is not considered mathematical flexibility since it is the direct use of a mathematical formula.

**Results of the grid activity**

In the second task, in which it is asked to calculate the area of a rectangle on a grid and its hidden interior, a total of six different resolutions are contemplated: application of Pick’s Theorem, decomposition of the complementary figure into right triangles, use of Pythagoras' Theorem, sum of the squares and triangles, sum of the squares and complementary triangles, and from the diagonal. In addition, as in the exercise described above, there are teachers in training who were not able to solve this exercise. The first two resolutions already appeared for the first exercise, while the other four came up in response to this task.

In the first of the resolutions addressed, students used Pick’s Theorem (see Figure 7). To do this, they had to draw the grid inside the rectangle, as it was hidden, and then consider this grid as an orthometric geoplane. The following steps are identical to those already explained for the first task: application of the formula $d + \frac{f}{2} - 1$.

![Figure 6](image1)  
**Figure 6.** Frequency of each resolution.

![Figure 7](image2)  
**Figure 7.** Student Resolution. Use of Pick’s Theorem.
The second of the resolutions analyzed had also been found in the first exercise. The students calculated the triangles outside the rectangle and subtracted the sum of all of them from the area of the outer square (see Figure 8). This strategy has been presented in students who did not solve the first task, and who could have replicated it in an almost analogous manner.

![Figure 8. Student Resolution. Calculating the area with complementary triangles.](image)

In the third resolution we find the use of the Pythagoras’ Theorem (see Figure 9). Although none of the students proved that the geometric shape was a rectangle, some considered it as such directly because of its visual appearance. They applied this theorem to determine the base and height of the rectangle in order to apply the formula they are most used to from the figure: base per height, $b \cdot h$.

![Figure 9. Student Resolution. Use Pythagoras’ Theorem.](image)

The fourth of the resolutions could have been made without any mathematical knowledge, because you need to know about the symmetry of the square. Since the colored rectangle is made up of full and half squares, it would be sufficient to count them (see Figure 10).
The same applies to the fifth of the resolutions. The students who chose this form of calculation counted all the squares on the grid and subtracted the exterior squares that were visible, without having to reconstruct the grid on the inside (see Figure 11).

The sixth of the resolutions offered by the teachers in training was not foreseen by the teachers of the subject, having to develop also the mathematical flexibility for its correction. In it, the rectangle is divided into two equal squares and the diagonal of the outer square is calculated. Later, the side of the square is determined, being this the third part of the diagonal. Finally, the area of one of the squares into which the rectangle has been divided is calculated and multiplied by two. See Figure 12.

Figure 10. Student Resolution. Calculating the area as the sum of squares and triangles.

Figure 11. Student Resolution. Calculation of the area with sum of squares and complementary triangles.

Figure 12. Student Resolution. Calculation of the area from the diagonal.
As was the case in the first exercise, with the square-frame geo-plan, the frequency of these resolutions is not uniformly presented. In this case, the highest percentage of students is found in those using the Pythagoras’ Theorem, 26.98%, followed by those using the Pick’s Theorem, 17.46%. The rest of the resolutions are given with a low incidence: 6.35% from complementary triangles, 4.76%, 3 people, with the sum of squares and triangles, 4.76% with the sum of squares and complementary triangles, and 1.59%, 1 person, with the calculation of the diagonal. Once again, a high percentage of students do not successfully complete the exercise, 38.1% despite the fact that it was designed for 5th and 6th grade Primary School students who participated in the Spring Contest². See Figure 13).

![Figure 13. Percentages of each resolution.](image)

In this exercise, spontaneous mathematical flexibility is considered in the calculation of area through the complementary triangles, as it already happened in the first exercise; and also that which involves the diagonal of the outer square in its development. In addition, with the latter, the subject's teachers also had to develop a mathematical flexibility, as mentioned above. The counting of squares and triangles (half squares) and the counting of complementary squares and triangles are not considered spontaneous mathematical flexibility, since they could be done without any mathematical knowledge. Neither is the resolution using Pick’s Theorem or Pythagoras’ Theorem

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² Mathematical Competition for 10-18 years old students
since the aim is to use a mathematical formula and not a reflection on its resolution, despite having to perform a greater number of calculations than for other options.

Comparison between the results of the two tasks

Despite the fact that we found two exercises of similar characteristics and difficulty - even the first exercise is a little easier - the results in both have been very different, as far as mathematical flexibility is concerned.

While in the first exercise two different resolutions have been observed, in the second one up to six different ways of reaching a correct and reasoned solution to the proposed problem have been counted. Furthermore, the two resolutions of the first exercise have also been perceived in the second one.

Not all students solved both exercises in the same way; in fact, while Pick’s Theorem was used in the first exercise by 41.27% of the students, this percentage was reduced to 17.46% in the second. This percentage reduction is justified because during the explanation of Pick’s Theorem in the classroom it was used with a problem similar to the first case, with the use of the orthometric geoplane and the interior of the visible polygon. Its use in both cases, despite the percentage difference, was a non-flexible resolution, even more so for the first case because it is an exact replica of an exercise already carried out.

Similar percentage is presented for the resolution from the complementary triangles: 7.94% in the first exercise and 6.35% in the second one. The steps to be carried out in both cases were similar, and with them the mathematical flexibility was developed.

In the case of the second task, students did perform the decomposition into simpler geometric shapes: 4.76%, 3 of them each, directly and 4.76% with the complementary ones. This strategy, moreover, has been presented in students who did not solve the first task, and who could have replicated it in an almost analogous manner.

With the characteristics of the first problem, for the calculation of the area of a quadrilateral it was not possible to apply the Pythagoras’ Theorem or the calculation of the diagonal, although there could be other strategies not contemplated by the teacher and the research team.

The high percentages of unsuccessful resolution of both problems are striking. It should be remembered that the level for which they are intended is 5th-6th grade Primary Education. Specifically, the first of the problems was not carried out correctly or was left blank for 50.79% of the students. Although the percentage in the second case is lower, it also reaches a high value, 38.1%. In both cases, these percentages are higher than those of any resolution strategy.

If we analyze the percentages together and contemplating the resolution strategies (see Figure 14), flexible or not flexible, we obtain that, while in the first exercise the students using the Pick’s Theorem and calculating the area of the complementary triangles represent 49.21% of the total, for the second, these same two cases represent 23.81%. From this we can deduce that at least 25.4% of students in the Primary Education grade are able to modify their resolution strategy, adapting it to the data and geometric forms provided. In addition, 12.69% are able to carry out a
strategy if any of the data or geometric forms vary, despite not being able to solve other similar problems.

![Sectorial chart second problem.](image)

**Figure 14.** Sectorial chart second problem.

**Conclusions**

In this research, we have been able to deepen our understanding of the spontaneous mathematical flexibility that students in the Primary Education grade have when they face an area calculation problem. The purpose of this theoretical framework is not only to address the fact that students correctly and justifiably perform a certain problem, but also that they are capable of adapting this resolution to each problem and innovate with the development of their answer, even going so far as to perform procedures not expected by their teachers (Xu et al., 2017).

Analysis of the responses of teachers in training shows that only in 7.94% (area by complementary triangles) of cases is there mathematical flexibility for the calculation of the area requested in the first problem, and 25.4% (Pick, complementary and diagonal triangles) of students are flexible for the second problem.

In addition, among all the individuals that make up the sample, we highlight the 12.69% that are capable of modifying the resolution strategy for the data and geometric shapes provided.

With the difference between the results of both exercises we can affirm that mathematical flexibility is encouraged depending on the type of problem requested. In the first case, it was an exercise similar to the explanation offered for Pick’s Theorem in the classroom; consequently, a good number of students, 41.27%, opted for this strategy, without having to make any additional reflection to that of the application of a previously memorized formula. In the second case, a reflection by the university student is further enhanced by not exactly replicating a problem in the
classroom, even if only a geoplane is modified by a grid and the visibility of the interior of the geometric form.

This fact should promote a reflection by active teachers on how statements condition resolution strategies and can inhibit flexible reasoning.

This study has some limitations, as already occurred in the analysis of mathematical flexibility for fractions in Lee's research (Lee, 2017). This happens because the problems posed have some particularities that can influence the strategies used by students, having to adapt, in a cyclical manner, the tests carried out on the students.

The research foresees as a future perspective to deal with the evaluation of the analyzed resolutions by other teachers in training, being able to analyze the mathematical flexibility that they develop as teachers and not only as students.

Acknowledgment

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References


Understanding the Hemingway measure of adult connectedness survey by utilizing data analysis

Ping Ye, Gildardo Bautista-Maya

ping.ye@ung.edu

University of North Georgia, USA

Abstract: This paper analyzes the dataset collected from students participating in the Boy With A Ball (BWAB) program, a faith-based community outreach group, through the Hemingway Measure of Adult Connectedness©, a questionnaire measuring the social connectedness of adolescents. This paper first approaches the data in the conventional method provided by the Hemingway website. Then it identifies which questions are strong determiners in deciding whether a student has completed the BWAB program or not. With the goal of utilizing the logistic regression, the set of questions to those only identified as significant in other methods is reduced. These methods include linear regression, decision tree, and random forest. The results are explained from a psychological perspective of social adolescent development.

Keywords: Hemingway scoring method, student characteristics, connectedness

1. Introduction

Boy With A Ball (BWAB) is a non-profit organization that works to make cities better places by reaching young people and equipping them to be leaders capable of transforming their communities. The BWAB program works in multiple locations, both across the nation and globally, intending to develop troubled youth and thus develop communities. These solutions include mentoring, faith-based camps, scholarships, and community development. BWAB relocated its global headquarter from St. Antonio Texas to Atlanta Georgia in July 2013. The authors have been worked with BWAB since 2017 to help analyze data and evaluate its mentoring program under the support of the MAA PIC Math Grant and the UNG LEAP into Action Grant. Given the record of the BWAB program, which includes increased academic performance and graduation rates for students who are part of the program, the program believes that getting these kids connected is working positively in satisfying Maslow's needs (Maslow, 1943) and overall

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http://www.hostos.cuny.edu/mtrj/
improving the student participants' future potential. The Hemingway Measure of Adolescent Connectedness survey is the first research-based measure of adolescent connectedness. The Hemingway was developed in response to the need for an effective way to evaluate the impact of a high school mentoring program. Utilizing data gathered through the Hemingway Measure of Adult Connectedness® questionnaire, administered by BWAB during 2010-2013, the authors analyze question importance through linear regression, decision tree, random forest, and logistic regression. Furthermore, the authors use the Hemingway scoring method to compare participants who have completed the program to those who have not to see which aspects of social connectedness separate the two.

The dataset contains the question answers for 220 ninth-grade students, now are called participants, who were referred through their schools to participate in the BWAB program. 38 of the 220 have participated in BWAB in the past, assigned a “Program” value of 1. 182 of the 220 have not completed the BWAB program, assigned a “Program” value of 0. For future reference, the implication of which group a participant is relevant in the sense that someone who has not completed the program still needs it, while those who have do not. Furthermore, the group assigned with a value of 0 included those who were new to BWAB as well as those who did not complete the program in its entirety. Those who have been assigned a “Program” value of 1 completed the questionnaire as a post-survey. In contrast, those assigned a value of 0 completed the questionnaire finished it as either a pre-survey.

Each survey question was answered as one of the following five categories: “Not at all true”, “Not really true”, “Sort of true”, “True”, “Very true”, and “Unclear”. Generally, “Not at all true” was assigned a score of 1, “Not really true” was assigned a score of 2, “True” was assigned a score of 4, while “Very true” was assigned a score of 5; however, if the question is worded in such a way as to be reverse scored, “Not at all true” was assigned a score of 5, “Not really true” was assigned a score of 4, “True” was assigned a score of 2, while “Very true” was assigned a score of 1. In both cases, “Sort of true” and “Unclear” were assigned a value of 3, whether it was graded reversed or not.

2. Experimental Section

2.1 Comparison of Category Means

By comparing the resulting means, the categories in which the groups coincide or differ can be visualized. Furthermore, the categories in which each group scores low or high on connectedness can be observed. As far as the Hemingway Measure of Adult Connectedness® questionnaire, the scope of the study will supersede past its intended purpose.
The authors begin by using the Hemingway scoring method, which consists of scoring the participants based on question categories. The 57 questions are broken up into ten different aspects of social connectedness: Neighborhood, Friends, Present-Self, Parents, Siblings, School, Peers, Teachers, Future-Self, and Reading. Upon finding the average score for each category, the authors determine whether the difference between those who have completed the program (1) to those who have not (0), and whether the measured difference is significant.

Question values were set to the Hemingway standard from 1 to 5 and the values of reverse-graded questions inverted. Thus, the mean score for each question was taken from each group. Questions were then grouped according to their categories and given a cumulative mean. Without loss of generality, question numbers ending in 1 were from into the category of “Neighborhood”, 2 from “Friends”, 3 from “Present-Self”, 4 from “Parents”, 5 from “Siblings”, 6 from “School”, 7 from “Peers”, 8 from “Teachers”, 9 from “Future-Self”, and 0 from “Reading”.

![Figure1: Mean Score between Categories](image)

Above Figure1 shows the visualized means between those who have completed the program to those who have not. As stated by the Hemingway manual, scores measuring below a score of 3.5 denote “low connectedness”, while scores at or above 3.5 denote “high connectedness”.

The following table1 summarizes the category means of each group with “high connectedness” being denoted in blue and “low connectedness” being denoted in gold. As can be seen, only one category is marked as denoting low connectedness, which is “Neighborhood”. Meanwhile, the
group that has not completed the program has four categories denoting low connectedness, which are “Neighborhood”, “School”, “Peers”, and “Teachers”.

<table>
<thead>
<tr>
<th>Has Completed</th>
<th>Neighborhood</th>
<th>Friends</th>
<th>Present-Self</th>
<th>Future-Self</th>
<th>Parents</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.982</td>
<td>4.351</td>
<td>3.912</td>
<td>4.158</td>
<td>4.013</td>
</tr>
<tr>
<td>Has Not Completed</td>
<td>2.981</td>
<td>3.639</td>
<td>3.667</td>
<td>3.653</td>
<td>3.672</td>
</tr>
<tr>
<td>Has Completed</td>
<td>Siblings</td>
<td>School</td>
<td>Peers</td>
<td>Teachers</td>
<td>Reading</td>
</tr>
<tr>
<td></td>
<td>3.926</td>
<td>3.991</td>
<td>3.737</td>
<td>4.114</td>
<td>3.553</td>
</tr>
<tr>
<td>Has Not Completed</td>
<td>3.579</td>
<td>3.372</td>
<td>3.493</td>
<td>3.467</td>
<td>2.522</td>
</tr>
</tbody>
</table>

Table 1: Category Means

After computing the mean of each category, the significance of each was determined. Although not every individual question is significant to the 5% confidence interval, the combination of multiple questions results in most differences in the categories being significant.

Even if a category has a noticeable difference, it does not necessarily imply that the value of the questions was significant in determining which group a participant was in. Although the ‘Reading’ category denotes a large and significant difference between those who have completed the program and those who have not; yet, no question from that category was found to be ultimately relevant in predicting whether a participant was in the program or not.

The results are particularly useful in terms of observing the dataset from a psychological aspect. We note that there is reason to believe in a difference between the social connectedness of those who have completed the program and those who have not; however, we must go beyond the Hemingway's given categories and isolate which individual questions matter most in determining whether a student completes the program or not.

2.2 Identifying Significant Questions

By developing models to select which questions are the best predictors of which group a participant is in, we can narrow the full range of data to a few critical questions. Furthermore, we can observe the category in which these questions originate, thus finding an aspect of social connectedness in which the program can focus its efforts.

One of the issues of using linear regression is that the model may "overshoot" and predict values that are above the maximum value or below the minimum value. As seen in the linear regression model, over half the entries were assigned a value below 0, even though as a categorical variable, it would never be anything less than 0.
2.2.1 Logistic Regression Model

The ideal model when predicting a dichotomous categorical variable such as the participant needs the program or not would be the logistic regression. Here, the logistic regression model is used to reduce the number of independent variables by removing questions that are not impactful in deciding which participant needs the program.

For all models, the same training set and testing set are used. The training set and testing set are split in the way that the training set contains 70% of the dataset and has a size of 154 while the testing set contains the remaining 30% and has a size of 66.

After applying the logistic regression with significance level 0.05, questions 34, 39, 45, 50, and 56 are statistically significant. The confusion matrix of the testing set is computed, and it shows the accuracy of the logistic regression model is 86.364%.

2.2.2 Decision Tree Model

The authors want to compare different classification models to eliminate the independent variables. The second model used is the decision tree. In a decision tree, the end goal is to assign the entry a probability of whether a participant needs the BWAB program or not. After applying the decision tree model, the end of each branch assigns a predicted probability based on the Hemingway survey responses to a few questions. The decision tree results are as the graph.

Figure 2: Decision Tree Model
The questions that affect the model the most are shown below. The questions with a “variable importance” of 5 or larger are used. Thus, questions 6, 8, 22, 24, 39, 47, 48, and 56 will be taken for using the decision tree model. Finally, the confusion matrix shows the accuracy of the decision tree model is 81.818%.

![Variable Importance Chart](image)

Figure 3: Variable Importance Chart

It is worth noting that while a question may not be present on the decision tree visual, it does not imply that the question is an irrelevant predictor. Furthermore, just because a question is currently on the decision tree does not mean that the question is a good enough predictor. For example, take the branches that split for a participant’s response to question 41. It does not matter which answer they may or may not have marked, if the participant reached that part of the decision tree, they would have been assigned a value of 0 through utilizing our testing set.

### 2.2.3 Random Forest Model

The third model used is the random forest, which is essentially creating multiple decision trees were using the aggregate to find the best predictors. The authors created a random forest of 500 trees to minimize the error. By using the random forest to predict the values on the testing set, it shows that the model predicted results with an accuracy of 90.909%.

As shown in the graph below, the ‘purity’ of each question is a measure of how influential a variable being to the model. While there is no explicit cutoff to say which questions are more telling, only the variables with purity higher than one are used here. Thus, questions 8, 20, 39, and 40 will be taken from the random forest model.
Figure 4: Questions versus Purity

3. Results

Through these three preliminary models, the best predictors from each model are taken to use for a refined logistic regression model. By doing so, the number of independent variables is reduced from 57 to 13. The following table contains the questions marked significant from each model, thus showing that some questions are marked significant in all three models.
Table 2: Significant Questions among Three Models

<table>
<thead>
<tr>
<th>Question</th>
<th>Logistic Regression</th>
<th>Decision Tree</th>
<th>Random Forest</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>•</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>•</td>
<td>•</td>
<td></td>
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<tr>
<td>20</td>
<td></td>
<td>•</td>
<td></td>
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<td>22</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>24</td>
<td></td>
<td></td>
<td>•</td>
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<tr>
<td>34</td>
<td>•</td>
<td></td>
<td></td>
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<tr>
<td>39</td>
<td>•</td>
<td>•</td>
<td>•</td>
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<td>40</td>
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<td></td>
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<tr>
<td>45</td>
<td>•</td>
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<td></td>
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<td>47</td>
<td></td>
<td></td>
<td>•</td>
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<tr>
<td>48</td>
<td></td>
<td></td>
<td>•</td>
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<tr>
<td>50</td>
<td>•</td>
<td></td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>•</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After applying the new logistic regression model for the above 13 survey questions, it shows that six questions out of the 13 candidates are statistically significant in determining whether a participant needs the BWAB program or not. Furthermore, the authors see whether the correlation between the question and a participant's “Program” value.

With the new logistic modelling complete, the significant questions in order of lowest to highest p-value are shown as the following: Question 8 (0.236%), Question 50 (0.262%), Question 22 (0.468%), Question 39 (0.806%), Question 24 (1.207%), and Question 56 (3.106%). The categories in which these questions came out of are “Teachers”, “Friends”, “Parents”, “Future-Self”, “School”. Two of those, “School” and “Teachers”, are categories in which those who have not finished the program received a mean score considered “low connectedness”, which those who have finished the program did not.
4. Discussions

Concerning the Hemingway Measure of Adult Connectedness© questionnaire, each group’s cumulative mean for a question category is measured and interpreted as “low connectedness” or “high connectedness”. For review, a category is interpreted as a sign of “low connectedness” if it receives a score being less than 3.5; conversely, a category is interpreted as a sign of “high connectedness” if it gets a score of 3.5 or higher.

From the previous mean comparison, the group of participants who have not completed the program obtained a lower measure of connectedness in all categories when compared to those who have completed the program. Notably, the scores received in the categories "School", "Peers", "Teachers" were marked as "low connectedness" in the group of those who have not completed the program. In contrast, those who have completed the program were marked as "high connectedness". These categories have a real-world connection in the sense that these categories are shaped by what the participant experiences at school. Additionally, the highest-scoring category for any mean is the score for ‘Friends’ from the group that finished the program, who had a mean score of 4.351. It indicates that friends may be more critical than siblings in the adolescence period, as the focus is on expanding relationships beyond the family.

From the refined logistic regression result, it shows how disproportionally represented the "Teacher" and "School" question categories are when compared to any other category on the survey. Out of the six questions found significant, three were from the categories "Teacher" and "School". Questions such as question 8 (“I care what my teachers think of me”) and question 50 (“I usually like my teachers”) stress the importance of teacher interactions in a student’s life, with question 8 questioning how much the participant values their teacher’s opinion and question 50 asking how the participant feels about their teachers. Relating to the mean comparison, question 22 ("Spending time with my friends is a big part of my life") highlights the importance of spending time with friends. Such friends may surround the participant at school, in the program, or in their neighborhood.

An interesting result from the logistic regression model is that questions 24 ("I enjoy spending time with my friends is a big part of my life") and 56 ("Doing well in school is important to me") have a negative correlation associated with them. This is to say, the more ‘True’ the statement was to the participant, the more likely they were to be considered part of the group which has not completed the program, thus implying that they still needed the program. Yet, if the participant's response is accurate, perhaps they were well socially connected such that they did not need the program to begin with.
A notable observation is the accuracy of our decision tree model and the random forest model. The accuracy obtained from performing the modeling on the testing set denotes that this accuracy is not a sign of overfitting, but rather, that there exist questions whose responses are strong determiners of which group the participant is in, thus reinforcing our motive of isolating these questions for future study.

While we focused on what was significant, consider the categories which were not significant, “Neighborhood” and “Siblings”. Both groups scored almost identically in the category of “Neighborhood” with those who have completed the program scored a 2.982 while those who have not scored a 2.981. Furthermore, these scores are considered a sign of “low connectedness” in Hemingway. While the striking similarity between the groups is puzzling, it is worth considering that the data was collected during 2010-2013, a time whether technology does not require participants to be physically associated in terms of the questions asked in the questionnaire. Finally, it’s worth noting that questions that reside within the “Sibling” category were the ones that were not filled out and had to be provided a substitute. As a result, many students have an unbiased score of 3 for questions in this category, helping explain the lack of significance in this category.

Without a doubt, there is more than meets the eye in any data analysis. Adolescents’ level of “connectedness” to family, school, friends, and self has been found to contribute to academic performance but also predict violence and substance use. Fortunately, the school environment directly influences students’ levels of connectedness such that connectedness appears malleable to school-based interventions. Though the group consisting of those who completed the program scored higher than those who have not completed the program, the authors cannot establish that participation in the BWAB program definitively caused this change; however, what the authors can say is that those students with low scores in certain categories could benefit from the program, i.e., a pre-survey could be used to identify the students who need the BWAB program. The authors would like to utilize the above data analysis tools to the BWAB Atlanta data of the year 2014-2020 for future study to discover the importance of connectedness for teaching and learning in multilingual, multiracial and multicultural Atlanta school environments.

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Effect of Digital Awareness on Mathematics Achievements at School to University Levels in Nepal

Bishnu Khanal¹, Shashidhar Belbase², Dirgha Raj Joshi³*

¹Mahendra Ratna Campus Tahachal, Tribhuvan University Nepal, ²Department of Curriculum and Instruction, College of Education United Arab Emirates University, Al Ain, Abu Dhabi, United Arab Emirates (UAE), ³Mahendra Ratna Campus Tahachal, Tribhuvan University Nepal & Nepal Open University

*Corresponding author: dirgha@nou.edu.np

Abstract: Digital awareness is necessary for mathematics teachers to use digital technologies, including ethical, cultural, leadership, and policy awareness in this 21st century. This study aimed to examine the interrelation of digital awareness of mathematics teachers with students’ achievement at schools to higher education levels in Nepal. An online survey was conducted among 399 mathematics teachers of Nepal and Mann-Whitney, Kruskal-Wallis, and multilevel linear regression were major statistical techniques used in the study. The findings indicated that most of the participants had digital devices, and the level of digital awareness was found to be high. The types of institution and teaching level were major contributing factors to determine the digital awareness, and developing and sharing cultural consequences are the main predictors of learners’ achievement.

Keywords: Ethical and policy awareness, cultural and leadership awareness, digital awareness, mathematics achievement, Nepal

INTRODUCTION

In a study, Bennison and Goos (2010) stated that teachers' inclination to professional development encourages them to innovate new approaches to improve student learning of mathematical concepts. In this context, teachers' use of technology in innovative ways
fundamentally changes the content and learning process (Lynch 2006). For effective and innovative use of technological and other mathematics tools, teachers should have competencies of such tools (e.g., digital tools, computers) and mathematical content knowledge to utilize an integrated form of knowledge for effective teaching of subject-specific content (Campbell 2003). Therefore, teachers' technological, pedagogical and content knowledge (TPACK) impacts how proficient and skillful the teachers are in selecting digital technologies to represent, broaden, and connect mathematical activities for profound learning and higher-order thinking (Loong and Herbert 2018; Mishra and Kohler 2006). The use of technological and digital tools in education in general and mathematics education, in particular, has been increased in the Australasian region. For example, Australia has emphasized using digital tools to teach and learn mathematics at the school level (AAMT 2014). Likewise, the use of digital tools in Singapore schools has been emphasized with a "special computer ownership scheme" (UNESCO 2014, p. 18). Nepal also unveiled Digital Nepal Framework 2019 as a significant step to apply new technology and digital tools in education and other sectors (MoECIT 2019).

Teachers are considered as change agents of society (Badley 1986; Bourn 2015). Hence they should have knowledge and skills of innovations, creations, and adoption (Kovacs 2017; OECD 2016) in their discipline. Teachers also should have skills to motivate others towards correct and ethical use of technological resources. Mobile, computer, TV, radio, and the Internet are core ICT infrastructures (UN et al. 2005) at this age. Technologies are being advanced day by day. Misappropriation of digital resources may cause hostilities, aggression and violence, sexual abuse, commercial exploitation (UNICEF 2017). Hence the safe use of such technology is a necessity for all global communities.

Misuse of technology is a growing concern in all sectors, including education and teaching-learning. One of the most misuses of technology is cybercrime. There are several provisions to combat cybercrime due to the misuse of technology. Technological advances have been changed and affected every sector of human life, from food, medicine, transportation, education, research, and entertainment (Al-Saqqa et al. 2014; DoCE 2019; Elsobeithi and Abu Naser 2017; Sutton 2013). People of all ages and professions are affected more or less by modern technology, such as digital gadgets and tools. This immense transformation in life and thinking through digital technology has raised concerns about identity, safety, privacy, and digital content publicity. These concerns are genuine in the proper use of digital information at both personal and public levels. In this context, one can raise digital awareness questions in schools and higher education institutions in Nepal. Are school and university teachers and students aware of their privacy and internet security in Nepal? Do they care about digital footprints? Do teachers have an awareness of the responsible use of digital devices and online materials? This paper discusses the issues around these questions, in general, to look at the overall perception of the teachers with a focus on digital or technological tools such as mobile phones, computers, and television. More importantly, it explores how mathematics teachers' digital awareness level impacts on students' achievement or performance in mathematics from school to higher education in Nepal.
The Internet has been an essential means of digital technology and information in Nepal in recent years. There were only about 7000 Internet users by 1999 (Pradhan 1999), and this number reached nearly 30,000 in the year 2006 (Kasajoo 2006). It grew further to 19.7% of the total population in 2016 and 34% in 2017 (The World Bank 2020). Internet broadband penetration increased sharply from 2016 to 2017. It penetrated 63% of the total population in the year 2019 (MoCIT 2019). ITU report showed that 21% of individuals using the Internet in Nepal.

In contrast, that rate is 44.3% in Asia & Pacific and 48.6% in the world, a household with computer access is 14%, 38.9%, and 47.1%, and the 3G coverage population is 54.1%, 91.3%, and 87.9% in Nepal, Asia & Pacific and in the world respectively (ITU 2018). This status shows that the position of Nepal is comparatively low in comparison to others. The study is concerned with mathematics teachers from primary to university level only. This data included both broadband wired and wireless and mobile Internet usage. The growth of the Internet and smartphone penetration in the country certainly demonstrates a growing awareness of ICT use in daily life, business, communication, and education, among many other activities. However, the government lacked a firm national policy and guidelines to regulate Internet technology's proper use for various sectors until 2019, despite some policies and prior guidelines for ICT and the Internet.

For the first time, Nepal unveiled the 2019 Digital Nepal Framework to adopt technological advancement in all government, public, and private entities (Ministry of Communication and Information Technology [MoCIT] 2019). The vision of Digital Nepal, the framework emphasized the Internet penetration for economic growth, innovation to solve many challenges and tap into the global economy's opportunities amid the development of both neighboring countries- India and China through digital connectivity (MoCIT 2019). The framework has charted a broad vision with one nation, eight potential sectors with eighty digital initiatives to leverage disruptive technologies and the country's socio-economic transformation. One of the eight sectors in the digital framework is education as a key priority area to develop connectivity, access, and empowerment of all stakeholders. Among the eighty digital initiatives identified, the first one focuses on establishing a reliable and essential Internet service by taking the lead by developing 5G networks to leverage all other sectors, including education. The other initiatives have links with Internet connectivity in general. The education sector has eight initiatives integrated into the framework. The first initiative is to develop smart classrooms. The second initiative is to develop OLE Nepal 2.0 within this initiative. An online portal, E-Path, and E-Pustakalaya have been designed to provide access to educational resources to the students, teachers, and parents. It also supports teacher training and developing the e-infrastructure of educational institutions in Nepal (see http://www.olenepal.org/). The other initiatives are – rent-a-laptop program, EMIS 2.0, centralized admission system, biometric attendance and CCTV, and mobile learning centers (MoCIT 2019). ICT regulation scores of Singapore, Japan, and Australia were high in Asia-Pacific; however, that score is low in Nepal's context (ITU 2020).

For effective use of digital resources, every teacher should know local, national, and international ICT policies related to their fields. They should design activities and lessons with the technological tools to implement their strategies for their instructional activities. The policies also
provide guidelines for the use of technology in their teaching-learning activities. Nepal has initiated several other policies to support digital technology. For example, the National ICT policy 2015, the National Broadband Policy 2015, and the Electronic Transaction Act 2008 are some policies to guide the development and use of digital technology in Nepal (MoCIT 2019). These policies and other guidelines by the government are helping to initiate digitization in public services and education. In this line, the ICT Policy 2015 aimed to increase ICT accessibility to a broader public arena through ICT infrastructure development, ICT industry promotion, e-Governance, and Human Resources Development in the field of ICT (MoCIT 2015). Likewise, the Broadband Policy 2015 aimed to extend the connectivity, public-private partnership, universal access, and minimize the digital divide and increase the access and coverage of broadband Internet service throughout the country by wireless or wired Internet connections (MoCIT 2015). The Ministry of Communication and Information Technology has promulgated Geo-Satellite Policy 2020 to connect Nepal to space for reliable and high-quality information services to reduce the dependency on other commercial Geo-Satellites communication services. This initiative is also related to digital content development and distributing those contents through the satellite networks at a faster and cheaper rate (MoCIT 2020). These are some current developments in Nepal in ICT and digital development initiatives that need a higher level of awareness and preparedness for a better future.

Ethical use of digital resources is related to individuals' moral values (Hamiti et al. 2014; Hoq, 2012). Different violence arises from the improper use of technology (Mancini and O'Reilly 2013; Martin 2010). Digital stalking, digital hate, threats of the virus, cyber terrorism, and digital spying (Mitra 2010) are major challenging issues. There are several laws of the act for computer-related crimes like the Electronic Transaction Act 2008 in Nepal, which was first introduced and implemented by the Government of Nepal. The act has a provision against the offense of relating to computers in Chapter 9 under this document. The act defined some illegal activities as cybercrime, threat, attract and abuse through the use of the Internet and provision of punishment with imprisonment not exceeding five years and with a fine not exceeding Rupees 200000 (2705 USD) based on the nature of activities (Giri 2019; GoN 2008). Hence, training and awareness programs should be given to all general public for safely using such resources (Olcott et al. 2015). Ethical and intellectual property rights-related content should be included in the curriculum (Bandara and Ellepola 2019). Hence every teacher and student should be aware of different digital resources (ISTE 2017) as software, online resources, and other e-resources (Buch et al. 2017). Digital awareness of the mathematics teacher may also support how they utilize technological tools, such as smartphones, computers, iPods, tablets, and tools available for computational and representational activities promoting mathematical thinking (English 2018). Teacher awareness and skills in using the varieties of digital tools may also support designing mathematical tasks for analyzing data and computing in a context that helps students "reinvention of mathematics by students themselves" (Langrall et al. 2011).
The digital awareness of people, at different sectors in general and students and teachers in particular, plays a significant role in the quality use of the ICT tools. Policy awareness is a fundamental and necessary skill for 21st-century teachers (UNESCO 2013). The levels of public awareness and engagement in digital technology and tools have been explored in a few studies in Nepal. For example, Regmi (2017) studied Internet connectivity and the quality of mobile phone access to the Internet at four places in Nepal – Panauti, Tangting, Changu Narayan, and Kathmandu. The studied sites were Cyber Cafes, Libraries, Schools, and University Campuses. Regmi (2017) observed the different ways to connect broadband subscriptions through dialup, wireless modem, cable modem, and ADSL. The findings of the study reported a low quality of Internet connectivity at educational institutions and public places. Although voice quality was satisfactory on the mobile phones in those places, the Internet quality on phones was not satisfactory for the users (Regmi 2017). The quality and variations of the Internet connection have not yet reached all people in Nepal due to geographical terrain and economic classes (Acharya 2016). These access issues have created a digital divide between people living in urban areas and the rural, remote areas that have affected experience and awareness toward the use of the Internet for education and other purposes (Acharya 2016). The interruption or lack of consistent internet access to Nepal's rural and remote areas is also caused by problems in power supply, bandwidth, lack of technical support, and, most importantly, lack of awareness of the people (Kasajoo 2006).

Digital awareness is related to having basic literacy skills in digital tools, such as smartphones, tablets, iPads, and computers. The basic skills of these devices, together with the Internet, provides students and teachers greater access to online resources for teaching-learning (Abbas, Hussain, and Rasool 2019). These skills influence students' academic performance (Amiri 2009; Lopez-Islas and Jose 2013). In the Research New Zealand report, 80% of school principals agreed that digital technologies positively impact students' achievement (Johnson, Maguire, and Wood 2017). Similarly, a US study reported that digital technology's use increased students' achievement in mathematics (Brasilé et al. 2016).

Similarly, there are views that mobile technologies have a significant impact on students' awareness and achievement through a higher level of creativity and problem-solving skills (Sung, Chang, and Liu 2016). Likewise, game-based learning has been a new trend to engage students in meaningful learning. For example, Kahoot has been one of the popular game-based learning tools that may have a positive influence on students' learning and classroom activities through enhanced thinking and awareness (Wang and Tahir 2020).

Despite the growth in the use of the Internet and digital-online tools for various purposes, there is still a lack of research that has documented the students' and teachers' digital awareness and how such awareness may affect students' achievement in mathematics (or other subjects) in Nepal. Therefore, we planned to conduct this study during the COVID-19 pandemic when all educational institutions (schools and universities) were closed for face-to-face classes. Some universities and schools were offering online and distance learning for the students. Students were using several online platforms, apps, and software for learning purposes. However, the effective use of these resources depends on their digital awareness, including ICT competence, to access...
various learning resources. The objective was to examine digital awareness of students' achievement in mathematics from school to university. The research question was: what is the awareness level of mathematics teachers in using digital technology? Furthermore, digital awareness affects students' achievement in mathematics from the viewpoint of teachers? In the rest of this paper, we outlined a literature review, theoretical framework, methodology, results, discussion, and conclusion.

LITERATURE REVIEW

We reviewed a few selected literatures from 2007 to 2020 on technology in teaching and learning and their impacts on students' engagement and quality of learning and performance in different subject areas. Lei and Zhao (2007) investigated to examine how students use technologies, what technology uses are popular among students, and what technology uses are useful for increasing student academic achievement. The data was collected from the seventh and eighth-grade students and teachers in highly productive technology equipped middle school in the state of Ohio in the USA in the academic year 2002-2003, employing survey and interview methods. The participants for the study were 207 for the pre-test and 231 for the post-test survey. However, 177 students participated in both the survey. The survey questionnaire to the student participants consisted of multiple-choice and four-point Likert scale items along with the question to report their GPA as academic outcomes. Besides the survey, a semi-structured interview was conducted with ten teachers and nine students to gain opinions and concerns on how students utilized technologies and for what aims. The quantitative data were analyzed using Correlation, T-tests, and ANOVA tests to find the association between technology uses and change in student GPA. In contrast, the interview data were coded according to the research questions of the study. The outcomes demonstrated that the amount of technology utilizes alone is not fundamental to student learning. Additionally, when the nature of innovation use is not guaranteed, additional time on PCs may cause more mischief than an advantage. Students' progress in GPA indicated that the innovation utilizes positively affected students were those identified with explicit branches of knowledge and concentrated on student development. Investigation results found that the use of innovation that had a positive effect was not well known. However, some utilization of some procedures was the least, and they were again utilized.

Barkatsas, Kasimatis, and Gialamas (2009) conducted another investigation to examine the intricate association between students' mathematics confidence, confidence with technology, attitude to learning mathematics with technology, active engagement, and behavioral engagement, achievement, gender, and year level. The study consisted of the 1068 year nine and year ten students from 27 randomly selected state co-educational schools in Metropolitan Athens, Greece. These students varied from upper-middle to low socio-economic status. The data was collected using the instrument the mathematics and technology attitudes scale (MTAS). The data were analyzed using exploratory factor analysis, correspondence analysis, cluster analysis, MANOVA, and chi-square test statistic. The results depicted that boys communicated more positive perspectives towards mathematics towards the utilization of technology in mathematics, contrasted with girls. It indicated that high mathematics accomplishment was related to significant levels of
mathematics confidence, firmly positive degrees of affective engagement, and behavioral engagement. The achievement was related to increased trust in utilizing technology and an inspirational mentality to learning math with innovation. A low degree of mathematics achievement was related to lack of confidence, low affective engagement, less trust in technology, and a negative demeanor to learning mathematics.

Parishan, Jafari, and Nosrat (2011) examined the effect of technology-based learning in biology on students' academic achievement. The study used a quasi-experimental pre and post-test design. The population of the study was 5240 female students at Khomeini Shahr junior high schools. They applied cluster sampling to assign students in the control and experimental groups of 27 junior high school students selected randomly in 2009-2010. The experimental group was taught using the technology-based active learning method, and the control group was taught using the traditional lecture method. The researchers used pre and post-tests to assess the effects of two instructional approaches. The achievement data were analyzed using descriptive statistics such as mean, standard deviation, and inferential statistics the covariance analysis (ANCOVA). The results demonstrated that the students' achievement in the experimental group was higher than the control group, and the difference was significant at the 0.05 level of significance. The results of the study confirmed that there was no significant difference between the groups' performance in terms of their family characteristics (parents, number of siblings, and economic status) (p>0.05).

Harris, Al-Bataineh, and Al-Bataineh (2016) examined whether one to one (1:1) technology affects students' academic achievement and motivation to learn in an elementary school in Illinois. The study employed a quantitative experimental design to collect the data from 4th-grade students from two different classrooms in the same school. The school had a low-income rate of 84.3%, and most of the students were African American, and the least were Asians. The 1:1 technology was implemented in the school (experimental group) of 25 students, whereas 22 students were in the traditional classroom (control group). The data collection instruments were based on tests from the Discovery Education Assessment (Math) and the school's attendance records for each month. The results showed that performance of students who were in the 1:1 implementation achieved higher in three sets of topic tests (in groups A and B) whereas the traditional group achieved higher in the rest of the other three sets of tests (Group C), indicating a mixed result of the study. Although 1:1 technology could be a factor in student achievement and motivation to be at school, the effect was not conclusive as the control group also achieved higher in group C. Harris et al. (2016) also concluded that 1:1 technology might be the impetus required for school districts to enable their students to accomplish at more significant levels.

Al-Hariri and Al-Hattami (2017) investigated the impact of technology usage on student learning achievement in the physiology courses at five colleges of health sciences in Dammam. The data was collected using a survey questionnaire with a five-point Likert-scale. The questions were related to their use of technology and devices. The information was collected in an online survey from 219 second-year students studying physiology courses in five colleges of different health science disciplines at the University of Dammam. The descriptive statistics and Pearson correlation coefficient were used to find the frequency and relationship between technology and
academic achievement in physiology courses. The results revealed a significant correlation between the use of technology and student achievements in their respective classes. The researchers found that the most-used devices were laptops (50%), and the least used devices were desktop computers (only 0.5%). Al-Hariri and Al-Hattami (2017) further stated that technology plays a significant role in promoting learning through many practical instructional approaches, such as self-directed, independent, and collaborative learning. They found that most students preferred using laptops instead of traditional desktop computers for their projects, assignments, and other academic activities.

Salvo, Shelton, and Welch (2019) examined factors that added to the effective fulfillment of online courses for African American male college understudies. The study employed a phenomenological approach. The researchers collected qualitative information from ten purposively selected male undergraduate students. They had finished an online college course from an accredited public university in the southern region of the United States. The information was collected utilizing a semi-structured interview with the participants regarding their academic achievement techniques and learning encounters. The results demonstrated several factors for motivation for online courses. For example, economic, educational, continuity, and unbiased environment, to name a few. The participants preferred self-teaching in some subjects that did not require face-to-face interaction with the teachers. In other courses, such as mathematics, they preferred face-to-face instead of online classes. The participants also viewed that they could have their voices without biasedness because everyone could participate in the discussion. There was no fear or anxiety in the online class because they could participate in the interaction. They liked to learn at their own pace and without time-pressure.

Zgheib and Dabbagh (2020) conducted a study to investigate how experienced faculty utilize web-based social networking to help learn exercises in their courses. The study focused on analyzing the sorts of social media learning activities (SMLAs), their plan, the intellectual procedures they support, and the kinds of information that understudies take part in while finishing SMLAs. The study employed a qualitative approach with quantitative results using multiple case-study designs. The researchers conducted various interviews with six faculty members who implemented Second Life based virtual class activities in their courses. The events were held in the sessions for two years in a public higher education institution in the mid-Atlantic region of the US. The study also gathered information from the observation of 115 students' course-related social media posts. The data analysis was based on Bower et al.'s conceptual framework for Web 2.0 learning design, Bloom's Taxonomy of Cognitive Domain, and Krathwohl's (2002) Knowledge Dimensions. The outcomes demonstrated that internet-based second life could bolster student learning and advance various intellectual procedures. The findings additionally uncovered that accomplished personnel could select appropriate Second Life instruments depending on their innovation highlights or their ubiquity in the field of study. They suggested coordinating a few media sources in the plan of a solitary SMLA. Zgheib and Dabbagh (2020) proposed that accomplished faculty who utilized the Second Life, explicitly wikis and sites, used them as Learning Management Systems.
The abovementioned studies outlined the impact of digital technology on student achievement, mathematics confidence, positive attitude, and social engagement in learning through web-based tools. Government policies have emphasized the use of technology and digital tools for teaching and learning mathematics (e.g., MoCIT 2019). The position statements of professional organizations, such as the National Council of Teachers of Mathematics (NCTM 2011), The Australian Association of Mathematics Teachers (AAMT 2014), and Joint Mathematical Council of the United Kingdom (JMC 2011) also suggested the use of digital tools in the classrooms.

Methodology
Design

A cross-sectional survey design was adopted in this study. The data were collected from the mathematics teachers of all levels of schools to the universities in Nepal. An online survey questionnaire was designed by using Google Form. The survey link was communicated and shared through social media, teacher training webinars, workshops, professional development webinars, and mathematics teachers' organizations in Nepal during the COVID-19 pandemic from May to July 2020. During the period, 399 mathematics teachers responded to the online survey from the seven provinces.

Variables

The outcome variables in the study were-- ethical awareness, policy awareness, cultural awareness, and leadership awareness under digital awareness and average scores of students in mathematics taught by the teachers in the last tests. All the items in the questionnaire were associated with the digital awareness of the respondents. The questionnaire had a five-point scale form of very true of me, true of me, neutral, not true of me, and not at all true of me. Additionally, marks of mathematics were calculated by the average of internal assignment or term examination marks and an average score of the final examination. Besides, a description of all items associated with all digital awareness has been presented in Fig. 1.
Fig. 1 Conceptual framework of digital awareness

There are twelve confounding variables—gender, experience, age, qualification, academic background, type of institutions, job type, teaching level, laptop, computer and TV, and ICT training as the teachers’ characteristics. The variable of having a mobile was excluded because all of the teachers (100%) under the study had mobile phones. The proportion of participants by gender was female 9.5% and male 90.5%. Participants’ distribution based on experience was 41.1% having less than ten years of teaching, and 58.9% had more than ten years of teaching experience. By age, 26.8% were 15-29 years, 59.2% were 30-44 years, and 14% were 45 to 60. Among the participants, 25.8% had an Intermediate or a Bachelor’s degree, and 74.2% had a Master or higher degree (M.Phil. Ph. D.). Based on the academic discipline, 67.7% were from the education field, 10.8% were from humanities/management (10.8%), and 21.6% were from the science background. Types of institutions have three categories as governmental (55.9%), private (28.1%), and public (16%). Job type had two categories as permanent (53.9%) and temporary (46.1%) based on institution and government rule. The teaching level had three categories as basic (grades 1-8) level (21.8%), secondary (grades 9-12) level (67.2%), and university level (11%) as per the rule of the government of Nepal. The participants having laptops were 81%, desktop computers 68.9%, and TV 49.9%. About 42.9% had no training, and 57.1% of the participants had some training in ICT or computer.
Research Instrument

A self-constructed online tool named "Digital Awareness Measurement Scale" was used for data collection. The instrument consisted of 13 Likert-scale type items with five-point ratings- 'not at all true of me,' not true of me,' neutral,' 'true of me,' and 'Very true of me.' Cronbach's Alpha reliability score of the tool was 0.94, which was very high. The instrument was shared with two survey design experts and two teachers to get feedback on clarity, suitability, and ease of understanding the items to establish content validity.

Analysis

The participants' responses were retrieved from the Google Form and imported to the IBM SPSS 26 for statistical analysis. The ratings in the items were coded from 1 = 'not at all true' to 5= 'always true,' respectively, to quantify the data. The mean scores of each item were calculated from the ratings, and composite scores were calculated from the groups of items within the four categories-- digital, ethical, cultural, and policy awareness. The composite values were the data into scale form. The assumption normality of these categories was tested by Kolmogorov-Smirnov (K-S) and Shapiro-Wilk (S-W) tests. The test result showed significant results for all categorical variables at a 95% confidence interval. Hence non-parametric statistics like Mann-Whitney and Kruskal-Wallis tests were used to test the significance of digital awareness results with respect to all socio-demographic characteristics (Cohen et al. 2007). Pearson correlation was used to measure the relationship between digital awareness and mathematics achievement scores. Additionally, a multiple binary regression was used to estimate the effect of digital awareness and achievement in mathematics.

Results

Table 1 shows that all mathematics teachers have at least one digital device. All the 399 participants have a mobile device (100%), which is the basic need of the 21st-century community. The majority of the participants have a laptop (81%), half of them have TV (50.1%), and only around one-third have computers (31.1%). The majority of the teachers are using digital devices (61.7%-90.4%) less than 3 hours/day, and around half (46.8%) to four-fifth (79.3%) of them are using such resources for more than five years.

Table 1 Availability and using the status of digital resources (n=399)

<table>
<thead>
<tr>
<th>Devices</th>
<th>Yes (%)</th>
<th>Hours of using digital devices</th>
<th>Years of using digital devices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>&lt;3 hours</td>
<td>3-6 hours</td>
</tr>
<tr>
<td>Mobile</td>
<td>399(100)</td>
<td>234(61.7)</td>
<td>105(27.7)</td>
</tr>
<tr>
<td>Laptop</td>
<td>323(81.0)</td>
<td>201(62.6)</td>
<td>90(28.0)</td>
</tr>
<tr>
<td>Computer</td>
<td>124(31.1)</td>
<td>129(85.3)</td>
<td>22(13.5)</td>
</tr>
<tr>
<td>TV</td>
<td>200(50.1)</td>
<td>170(90.4)</td>
<td>17(9.0)</td>
</tr>
</tbody>
</table>
The coefficient of variation (CV) predicts the items' accuracy, and the items having comparatively less CV seems to be better. Hence, along with the thirteen items under digital awareness, cultural understanding of global society has the lowest CV (0.18). Therefore, this item is more robust than others. The level of items has been determined in three categories based on the mean score of the items as high (3.67-5), medium (2.34-3.66), and low (below 2.34) (Fig. 1).

However, that level found to be medium among female (Mean=3.57), 45-60 years age grouped (Mean=3.65), humanities/management streamed background (Mean=3.64), basic (class 1-8) level (Mean=3.59) and university (Mean=3.49), not having a laptop (Mean=3.57) and did not participate any training (Mean=3.62) mathematics teachers in policy awareness and based on all remaining socio-demographic variables that level found to be high. Additionally, the result is statistically significant based on types of institution, teaching level, and those having laptops in ethical awareness in favor of private and secondary school teachers and those having laptops, respectively. Types of institutions and job types have a significant result on leadership awareness in support of private and temporary school teachers with their counterparts. Teaching level and status of taking ICT training have significant results on policy awareness in favor of secondary and trained mathematic teachers, respectively (Table 2).

### Table 2 Status of digital awareness with respect to different socio-demographic characteristics (n=399)

<table>
<thead>
<tr>
<th>Socio-demographic characteristics</th>
<th>Number of Respondents (%)</th>
<th>Ethical awareness</th>
<th>Cultural awareness</th>
<th>Leadership awareness</th>
<th>Policy awareness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gender</td>
<td></td>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>Female</td>
<td>38(9.5)</td>
<td>3.91</td>
<td>.67</td>
<td>3.93</td>
<td>.51</td>
</tr>
<tr>
<td>Male</td>
<td>361(90.5)</td>
<td>4.06</td>
<td>.70</td>
<td>4.00</td>
<td>.72</td>
</tr>
<tr>
<td>Experience</td>
<td></td>
<td>p-value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&lt;10 years</td>
<td>164(41.1)</td>
<td>4.08</td>
<td>.72</td>
<td>4.00</td>
<td>.76</td>
</tr>
<tr>
<td>≥ 10 years</td>
<td>235(58.9)</td>
<td>4.04</td>
<td>.68</td>
<td>3.98</td>
<td>.67</td>
</tr>
<tr>
<td>Age</td>
<td></td>
<td>p-value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15-29 years</td>
<td>107(26.8)</td>
<td>4.04</td>
<td>.66</td>
<td>3.98</td>
<td>.78</td>
</tr>
<tr>
<td>30-44 years</td>
<td>236(59.1)</td>
<td>4.07</td>
<td>.70</td>
<td>4.00</td>
<td>.69</td>
</tr>
<tr>
<td>45-60 years</td>
<td>56(14.0)</td>
<td>3.93</td>
<td>.74</td>
<td>3.96</td>
<td>.66</td>
</tr>
<tr>
<td>Qualification</td>
<td></td>
<td>p-value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Intermediate/ Bachelor</td>
<td>103(25.8)</td>
<td>4.02</td>
<td>.68</td>
<td>3.96</td>
<td>.79</td>
</tr>
<tr>
<td>Master/ MPhil/Ph. D.</td>
<td>296(74.2)</td>
<td>4.05</td>
<td>.70</td>
<td>4.00</td>
<td>.68</td>
</tr>
<tr>
<td>Academic background</td>
<td></td>
<td>p-value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Education</td>
<td>270(67.7)</td>
<td>4.06</td>
<td>.66</td>
<td>4.00</td>
<td>.70</td>
</tr>
<tr>
<td>Humanities/management</td>
<td>43(10.8)</td>
<td>3.96</td>
<td>.87</td>
<td>3.95</td>
<td>.73</td>
</tr>
<tr>
<td>Science</td>
<td>86(21.6)</td>
<td>4.02</td>
<td>.72</td>
<td>3.97</td>
<td>.74</td>
</tr>
<tr>
<td>Type of institutions</td>
<td></td>
<td>p-value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Governmental</td>
<td>223(55.9)</td>
<td>3.94</td>
<td>.76</td>
<td>3.93</td>
<td>.73</td>
</tr>
<tr>
<td>Public</td>
<td>64(16.0)</td>
<td>4.09</td>
<td>.59</td>
<td>3.99</td>
<td>.65</td>
</tr>
<tr>
<td>Private</td>
<td>112(28.1)</td>
<td>4.22</td>
<td>.56</td>
<td>4.11</td>
<td>.68</td>
</tr>
</tbody>
</table>
Since all the variables as ethical, cultural, leadership, and policy awareness and achievement, are parametric data; hence, the Pearson-product moment correlation was implemented for relationship. In Table 3, the significant correlation existed among all digital awareness related variables and mathematics achievements at 99% confidence level except as policy awareness with marks of mathematics; however, the relation is significant at a 95% level of confidence in this variable. The correlations are high among cultural and ethical awareness (r=0.77), leadership awareness with ethical and cultural awareness (r=0.7). The moderate correlation found in policy awareness with ethical (r=0.65), cultural (r=0.66), and leadership (r=0.68) awareness. However, the relation is low in achievement with culture and leadership, and negligible correlation existed in achievement with ethical and policy awareness (Burns & Dobson 1980, p. 247).

Table 3 Relationship between digital awareness with the average achievement of mathematics (n=399)

<table>
<thead>
<tr>
<th>Ethical Awareness</th>
<th>Cultural Awareness</th>
<th>Leadership Awareness</th>
<th>Policy Awareness</th>
<th>Achievement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>.77**</td>
<td>1.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cultural Awareness</td>
<td>.70**</td>
<td>.70**</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>Leadership Awareness</td>
<td>.65**</td>
<td>.66**</td>
<td>.68**</td>
<td>1.00</td>
</tr>
<tr>
<td>Policy Awareness</td>
<td>.19**</td>
<td>.22**</td>
<td>.20**</td>
<td>.12* 1.00</td>
</tr>
<tr>
<td>Achievement</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**p-value ≤ 0.05 (i.e. Significant)

*Correlation was significant at p= 0.01 level (2-tailed)
Hierarchical multiple regression was used to assess four digital awareness-related variables’ ability to predict the mathematics achievement score after controlling four dimensions of digital awareness variables. Preliminary analysis were conducted to ensure the assumption of normality linearly, multicollinearity, and homoscedasticity. The overall models explain 4% to 7% of the variance in the first to fourth models, which indicates that the model is poorly fit (Cohen et al., 2007, p. 538). The outputs generated from this analysis are in Model 1 to Model 4 based on digital awareness categories. Model 1 consists of ethical awareness-related items as independent variables and mathematics achievement as the dependent variable. The leadership awareness and policy awareness related items were added for Model 2 to Model 4, respectively. The model explains 4% of the variance in Model 1, which contained only ethical awareness-related items. After entry of ethical and cultural awareness related items in Model 2, the total variance was explained by 6%, F (7, 391) =3.29, p<0. By loading ethical, cultural, and leadership related items, the Model 3 generated, and the model explained by 7% variance, F(10,388)=2.7, p<0.05. The final model (Model 4) was calculated by adding all awareness related items, and models explain by 7% of the variance, F (13, 385) = 2.30, p< 0.05. The statically significant measure was calculated in develop and share cultural consequence under cultural awareness in Model 2 (beta=0.14, p<0.05, VIF=1.96), Model 3 (beta=0.15, p<0.05, VIF=2.16) and Model 4 (beta=0.16, p<0.05, VIF=2.20) with reference to Table 4.

Table 4 Multilevel binary regression on achievement with respect to digital awareness (n=399)

<table>
<thead>
<tr>
<th>Items with categories</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Beta</td>
<td>VIF</td>
<td>Beta</td>
<td>VIF</td>
</tr>
<tr>
<td>Ethical Awareness</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Safely use of digital resources</td>
<td>0.08</td>
<td>1.55</td>
<td>0.06</td>
<td>1.61</td>
</tr>
<tr>
<td>Ethical use of digital resources</td>
<td>0.01</td>
<td>2.17</td>
<td>-0.03</td>
<td>2.37</td>
</tr>
<tr>
<td>Copyright, intellectual property</td>
<td>0.06</td>
<td>1.97</td>
<td>0.05</td>
<td>2.00</td>
</tr>
<tr>
<td>Legal and ethical behavior in professional practices</td>
<td>0.08</td>
<td>1.82</td>
<td>-0.01</td>
<td>2.53</td>
</tr>
<tr>
<td>Cultural Awareness</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cultural understanding of global society</td>
<td>-0.01</td>
<td>2.04</td>
<td>-0.03</td>
<td>2.07</td>
</tr>
<tr>
<td>Develop and share cultural consequences</td>
<td>0.14*</td>
<td>1.96</td>
<td>0.15*</td>
<td>2.16</td>
</tr>
<tr>
<td>Promote digital etiquette</td>
<td>0.09</td>
<td>2.38</td>
<td>0.10</td>
<td>2.64</td>
</tr>
<tr>
<td>Leadership awareness</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Develop leadership by digital resources</td>
<td>-0.10</td>
<td>2.67</td>
<td>-0.09</td>
<td>2.77</td>
</tr>
<tr>
<td>Virtually participate in professional activities</td>
<td>0.05</td>
<td>2.04</td>
<td>0.06</td>
<td>2.06</td>
</tr>
<tr>
<td>Evaluate and share professional experiences with others</td>
<td>0.10</td>
<td>2.15</td>
<td>0.13</td>
<td>2.25</td>
</tr>
<tr>
<td>Policy awareness</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local and national level ICT policies</td>
<td>-0.03</td>
<td>3.21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ICT competency models of teachers</td>
<td>-0.10</td>
<td>3.34</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Develop digital technological models for classroom teaching</th>
<th>0.00</th>
<th>2.45</th>
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<tbody>
<tr>
<td>R²</td>
<td>0.04</td>
<td>0.06</td>
</tr>
<tr>
<td>ANOVA</td>
<td>3.76*</td>
<td>3.29*</td>
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</table>

*p-value ≤ 0.05 (i.e. Significant)

DISCUSSION

Leadership awareness may play a significant role in developing teachers' digital competency and awareness. It involves developing leadership in schools and higher education institutions to help teachers learn and develop their digital capability to apply in the classroom to create a digital-friendly classroom environment (Ottestad 2013). The goal of developing and achieving mathematics teachers' digital leadership and awareness of such leadership to foster a digitally rich classroom practice is possible through cooperative and distributed leadership development (Ottestad 2013). Digital technologies can support students' access high-quality mathematics, learning, teachers' awareness of the potential of using digital tools in promoting student engagement and professional development of teachers with an emphasis on equity (AAMT 2014). On the other hand, technological advancement in education and digital tools in the classroom has raised concerns about the teachers' ethical dilemma. Overexposure of students to the digital contents may distract them from the central aspects of conceptual knowledge and development of reasoning coherently, and even it may lead them beyond the social values. There is a danger of malware and viruses on the Internet that may damage persons and institutions' reputations. Also, there is a chance that some websites may publish inaccurate and misleading information that may affect both students and teachers (Northwest Missouri State University 2018). Therefore, digital awareness and technological innovation in education should be assessed in the light of social values and ethics (O'Brien 2020). In this context, it has been recommended that educational institutions and teachers should apply digital tools for "expressive and analytical purposes" in classroom teaching and learning of mathematics and beyond (Joint Mathematical Council of the United Kingdom [JMC] 2011, p. 5). For such development of technological digital tools and their appropriate uses in the classes should be guided by the policy at the school level to the national level. Students and teachers should be aware of the "changing norms of Internet usage …and consequences of their online actions" (Buchanan 2019, p. 2). The technology principal of NCTM (2000) standards, Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010), UNESCO (2028), and Department for Education (2019) have provided some policy guidelines about the use of the digital technology in the classrooms. The National Curriculum Framework for School Education in Nepal (MoECIT 2019) does include some provision of technological integration in the classrooms. Still, it does not provide a detailed policy guideline to the teachers and education leaders. It is high time to develop a direction and policy for the ethical use of digital tools in the classroom and outside by the teachers of all grade levels and higher education (Maggiolini 2014). The current study
explored digital awareness and its impact on mathematics achievement within the framework of four digital awareness dimensions—ethics, policy, culture, and leadership.

All mathematics teachers have at least one digital device; however, for most of them, the laptop was found to positively affect their ICT application and digital awareness. Around two-fifths of the teachers are using mobile and laptops for more than 3 hours per day, and more than half of them are using such devices for the last five years. This indicates that most of the mathematics teachers are digitally literate and able to use the resources for their different activities. The level of digital awareness was found to be high in all measured items and categories. One of the reasons for the high digital knowledge might be because the information was collected during the COVID-19 pandemic. The teachers learned about several digital tools and online resources during this period, which might have affected their digital awareness level. The mathematics teachers and their professional organizations conducted several technological skill enhancement programs through Zoom, Google Meet, Teams, Facebook groups, etc. However, that level was found to be medium among females 45-60 years of age. The digital awareness level was medium level among the humanities/management background and teaching at the basic (class 1-8) level. Digital awareness of teachers and students has implications from an "anthropological cultural point of view" (Combi 2016, p. 3). This awareness helps teachers and students play a positive role and develop their identity in the local, national and global arena by expanding shared knowledge, views, ideas, perceptions, and experiences using digital tools (Combi 2016). Technology and digital tools may support in enhancing students and teachers’ awareness of their cultural norms, values, and practices through greater access to the online resources related to cultural diversity, characteristics, and offerings with direct or indirect engagement through linking the cultural heritages in teaching-learning (Perez, Cabrera-Umpierre, Arrendondo, Jiang, Floch, and Beltran 2016).

Additionally, the result is statistically significant based on types of institution, teaching level, and those having laptops in ethical awareness in favor of private and secondary school teachers and those having laptops, respectively. Types of institutions and job types have a significant result on leadership awareness in support of private and temporary school teachers with their counterparts. Teaching level and status of taking ICT training have significant results on policy awareness in favor of secondary and trained mathematics teachers, respectively. A strong significant relation was found between all digital awareness categories with the achievement of mathematics. However, that relation was found to be low and negligible, which indicates that digital awareness is not highly contributing to the performance of the students. However, multiple linear regression showed that the cultural consequence under cultural awareness was the primary predictor of student achievement in mathematics. However, the coefficient of determination or the goodness of fit of the regression model is week as R² values are too low. Then, the predictability of the four models is weak. That means the teachers' digital awareness cannot accurately predict mathematics performance in terms of ethical awareness, leadership awareness, and policy.
awareness. The beta values are significant in models 2, 3, and 4 for one construct - develop and share cultural consequences within ethical awareness. These results indicated that the digital awareness of the teachers did not contribute to students' performance. One of the reasons for this unpredictability of students' mathematics achievement from teachers' digital awareness might be that digital awareness of teachers is a psychological state of teachers in terms of their preparedness to use the tools in teaching mathematics and their level of confidence in using the tools which not necessarily translated into the quality of students' mathematics learning.

CONCLUSION

Most of the Nepali mathematics teachers are digitally literate and have at least one digital resource or device. The digital awareness level on ethical, cultural, leadership, and policy was found to be high. Types of the institution, teaching level, and having laptops are contributing factors for ethical awareness, institution, and job types are contributing factors of leadership awareness, and job type and ICT training are contributing factors on policy awareness. The relation was found to be low and negligible on mathematics achievement with digital awareness; nevertheless, cultural consequences were found to be a predictor of mathematics achievement. However, this study's result was limited to 399 mathematics teachers' digital awareness in the context of Nepal, where there was limited use of digital tools in education before the COVID-19 pandemic. Enhanced use of digital prompted only due to schools, and higher education institutions closed their face-to-face classes due to the nationwide lockdown. The study results cannot be generalized to another context due to the difference in awareness levels and applications of teaching-learning tools. Hence, further investigation can be conducted among other subjects, students, and other social units, based on the country's diversity and large sample size. The finding of this research may inform the policymakers, teachers, researchers, and students to be safe using digital resources in this age. Additionally, the related stakeholders have to launch and implement some novel programs for digital awareness for teachers, students, and parents for the ethical use of digital tools for better learning and teaching.

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mathematics-education


“Perfecting” Leonardo da Vinci diagram

Ivan Retamoso

When looking at the famous Leonardo da Vinci’s diagram shown below

Diagram above thanks to Luc Viatour / https://Lucnix.be

We may ask ourselves, is it possible to have a square instead of the rectangle such that, the base of the square is tangent to the circle and the upper corners of the square are points that belong to the circle?
The answer to this question is: YES.

First, I present a method to construct this figure which we will call “Perfect da Vinci diagram”, this way I guarantee the existence of at least one figure satisfying the conditions stated before.

Method to construct a “Perfect da Vinci diagram”

1. In a Cartesian plane Draw a circle centered at (0,0) with radius equal to 5 units.
2. Draw a segment tangent to the circle at the point (0,−5) having length 8 units.
3. From the end points of the tangent segment to the circle raise two perpendiculars.
4. The perpendicular lines from step 3 will intercept the circle at two points, connect these two points with a segment.

The result of following the 4 steps is a “Perfect da Vinci diagram” and it is shown below:

Second, I look for a necessary and sufficient condition for the existence of a “Perfect da Vinci diagram”.

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http://www.hostos.cuny.edu/mtrj/
The sufficient condition is given by our method presented before in 4 steps, to search for a necessary condition I look for a relationship between the radius of the circle and the length of the side of the square.

In general, suppose we are given a “Perfect da Vinci diagram” constructed out of a circle of radius $R$ and a square with $L$ as the length of its side, see figure below.

![Diagram](image)

Then by the Theorem of Pythagoras we have that

$$B^2 + \left(\frac{L}{2}\right)^2 = R^2$$

From the Figure

$$B = L - R$$

Then

$$(L - R)^2 + \left(\frac{L}{2}\right)^2 = R^2$$

Expanding

$$L^2 - 2LR + R^2 + \frac{L^2}{4} = R^2$$
Subtracting $R^2$ from both sides of the equation, multiplying by 4 both sides of the equation and simplifying we get

$$5L^2 - 8LR = 0$$

Factoring out $L$ and eliminating it via division ($L \neq 0$ since it represents a length) we get

$$5L - 8R = 0$$

Equivalently

$$5L = 8R$$

Finally, we get

$$\frac{R}{L} = \frac{5}{8}$$

The above is a necessary condition for the “Perfect da Vinci diagram” to exist.

Conclusion:

A “Perfect da Vinci diagram” can be constructed using a circle and a square if and only if the ratio between the radius of the circle and the length of the side of the square is 5 to 8.

Remark 1:

Using our “Perfect da Vinci diagram” above.

If $L = 8$ units then

$$\frac{L}{2} = 4 \text{ units}$$

Using the ratio stated above

$$R = 5 \text{ units}$$

And from $B = L - R$

$$B = 3 \text{ units}$$

Which means that the shaded triangle is a Pythagorean $3-4-5$ triangle. This suggests to me that the Pythagorean numbers may be embedded in the proportions of the measurements of a human body, at least in average.

Remark 2:

If we are interested in comparing the perimeters of the circle and the square of a “Perfect da Vinci diagram”, without loss of generality we may let $L = 8$ units then $R = 5$ units
Doing the Computations.

Perimeter of the circle = $2\pi R = 2(3.14)(5) = 31.4$ units

Perimeter of the square = $4(8) = 32$ units

So, in any “Perfect da Vinci diagram” the perimeter of the square is always slightly larger than the perimeter of the circle.

Lastly, I superimposed the “Perfect da Vinci diagram” into the “Vitruvian Man diagram” and this is what happened, notice how the horizontally extended arms almost perfectly fit up to the wrists when properly chosen proportional circle of the “Perfect da Vinci diagram” is tangent to the circle of the “Vitruvian Man diagram” on the base.
Moreover, the Pythagorean 3 – 4 – 5 triangle mentioned before, is clearly now embedded in the proportions of an average human body as shown in the figure below.

![Pythagorean triangle in human body](image)

It is left to the reader to continue analyzing this “superposition” of figures to discover more properties that were not noticed when studying the proportions of an average human body.

**References**

   
   From Wikipedia, the free encyclopedia.
   

2. “Vitruvian Man” thanks to:
   
   Diagram above thanks to Luc Viatour [https://Lucnix.be](https://Lucnix.be)

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Spontaneous and reasoned approaches to the algebraic tiles’ factorization problem

Terence Brenner, Bronislaw Czarnocha

Hostos Community College

Abstract: In this paper, we present the spontaneous and reasoned approaches by a Physicist (B.C.) and a Mathematician (T.B). Section 1 is by B.C and section 2 is by T.B.

The events described below shows the interaction between two teacher-researchers during their class preparation. At the same it represents the first stage of the teaching experiment in the effectiveness of algebra tile method for understanding factorization through geometric pattern arrangement. During the next stage which will take place next semester they will introduce the method into instruction of intermediate algebra course. There is a striking difference between their methods which in general can be characterized as the inductive (BC) and deductive (TB) methods of reasoning.

Section 1 Aha!Moment
The problem found in one of the resource materials is stated below:

\[ 2x^2 + 3x + \square \]
Fill in the blanks by finding the largest and smallest integers that will make the quadratic expression factorable.

I did experience a gentle that is of low intensity Aha! Moment, when suddenly I felt empowered to go ahead; that is, to generalize from the separate concrete examples I have investigated into the general case, which helped me to solve the problem. I found one case without the hint but did not know the meaning here of looking for maximum and minimum integers that solve it, and how to look for them. By trial and error, I was able to find one case. The case I found was \( 2x^2 + 3x + 1 \) and, following the method of instruction for such a case that is to multiply “a” by “c” (ax^2 + bx + c) and to find its two factors whose sum is b, I found the integer +1 that solves the problem. I didn’t know what to do next, so I looked at the Hint to the problem, which suggested to use the visual model of the algebra tiles.

3 OpenMiddle.com
Hint
How would you represent this problem using visual model (such as algebra tiles)?

Then I recalled the method (Gningue, 2000) of the tiles on the example $x^2 + 3x + 2 = (x+1)(x+2)$; next, I modeled it on my own first solution, which was $2x^2 + 3x + 1$,

![Figure 0.1 Algebra tile model for $2x^2 + 3x + 1$](image1)

and then I started trying different possibilities, of which I could not find any by changing for different positive numbers $a, b, c$, even daring to employ fractions.

Only when I tried $(2x-1)(x+2)=2x^2 + 3x - 2$ to make the connection between its tiles model and the corresponding polynomial, and realized how it works, I had a gentle Aha! Moment, definitely noticeable on the cognitive level as generalization and on the affective level as empowerment. It allowed me to formulate the general condition of $(2x + a)(x + b)$ as the hidden analogy between examples I did, with the condition that $a+2b = 3$. Since $c = ab$, we get $c(b) = -2b^2 + 3b$, the analysis of this quadratic function helps to get the polynomials that work by assuming $b$ and getting $c(b)$, showing that $b=1$ is the maximum integer that solves it, while there is no minimum (Czarnocha and Baker 2020; Chapter 4)

![Figure 0.2 Algebra tile model for $2x^2 + 3x – 2$](image2)

Very proud of myself to have figure out how to deal with subtraction in the environment of tiles, and to deal in such a way that prompted generalization needed to solve the problem completely, I
carelessly handed the bag of color tiles to my office roommate, a mathematician. The next section describes what happened after my careless act.

Section 2 Some interesting ideas about factoring a trinomial using algebra tiles.

In this section we show what you need to know about algebra tiles and why and how you use the tiles to factor a polynomial. Using the geometric meaning of a rectangle, we show unusual ways to factor a trinomial based on the titles. I gave my colleagues eight algebra tiles consisting of one big red square that represents $x^2$, three green rectangles whose length represents minus three $x$ and four small yellow squares that represents negative four, i.e., $x^2 - 3x - 4$. I asked them to form a rectangle from these eight tiles. I also told them the red side of the big square represents positive $x^2$ and the black side represents negative $x^2$. The length of the red side of the rectangle represents positive $x$, the width positive one. The length of the green side of the rectangle represents negative $x$, the width negative one. The red side of the small square represents one, the yellow side represents negative one. Note: the actual dimensions are: big square 5cm x 5cm, rectangle 5cm x 1cm, and the small square 1cm x 1cm. Before looking at the examples my colleagues did, we showed the box method for multiplying $(x - 4)(x + 1)$ which is shown in figure 2.1

$$x + 1$$

<table>
<thead>
<tr>
<th></th>
<th>$x^2$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$-4$</td>
<td>$-4x$</td>
<td>$-4$</td>
</tr>
</tbody>
</table>

Figure 2.1

Then $x^2 + x - 4x - 4 = x^2 - 3x - 4$

A problem that everyone realized was, you cannot make a rectangle with these eight tiles. You are allowed to use more tiles if you follow condition 1.

Condition 1: You can add as many tiles to the original number of tiles as long as the number of positive tiles added of the same size is the same number of negative tiles added of the same size, that is you are adding zero.

I then asked them to multiply their left side with the top side. The examples we look at have a corresponding matrix that consists of $+$ and $-$ signs.

Our first example is Figure 2.2
The left column is \( x - 1 \) at the top is \( x - 2 \) (one red and one green rectangle cancel each other out). This represents \((x - 1)(x - 2)\). The related matrix has \( A_{11} = + \), \( A_{12} = - \), \( A_{13} = - \), \( A_{14} = - \), \( A_{15} = + \), \( A_{21} = - \), \( A_{22} = - \), \( A_{23} = - \), \( A_{24} = - \), and \( A_{25} = - \), that is \( A = \begin{pmatrix} + & - & - & - & + \end{pmatrix} \). \( A_{22} = A_{21} \times A_{12} = (-)(-) = + \), but \( A_{22} = - \). We have a contradiction, that is that the condition for multiplying sign numbers fails. This example required one more positive \( x \) and one more negative \( x \). The “longer” side is the top which uses five tiles that has length \( x - 4 \), the “shorter” side is on the left which uses two tiles whose length is \( x + 1 \), physically the top is the length, and the left side is the width, while mathematically the top is the width and the left side is the length.

Our second example is Figure 2.3
represents \((x + 1) (x - 4)\), the related matrix is \(A=\begin{pmatrix} + & - & - & - & - \\ + & - & - & - & - \\ + & - & + & + & + \end{pmatrix}\) When we check, \(A_{22}, A_{23}, A_{24}, A_{25}\) all follow the conditions for multiplying sign numbers. For example, \(A_{23} = A_{21} \times A_{13} = (+) (-) = -\) and \(A_{23}\) is negative. We have our first candidate for the solution of the problem.

Our third example is figure 2.4

![Figure 2.4](image)

This represents \((x + 2) (x - 5)\). The related matrix \(A=\begin{pmatrix} + & - & - & - & - \\ + & - & - & - & - \\ + & - & + & + & + \end{pmatrix}\)

When we look at \(A_{34} = A_{31} \times A_{14} = (+) (-) = -\), but \(A_{34} = +\), a contradiction again.

Our fourth example is figure 2.5
This represents \(x (x - 3)\) with the related matrix 
\[
A = \begin{pmatrix}
+ & - & - & - \\
- & - & - & - \\
+ & - & - & +
\end{pmatrix}
\]

As in example 1, we have the same contradiction, that is \(A_{22} = A_{21} \times A_{12} = (\neg) (\neg) = +\), but \(A_{22} = -\).

Our fifth example is figure 2.6
This represents \((x + 3)(x - 6)\) with the related matrix
\[
A = \left(\begin{array}{ccccccc}
+ & - & - & - & - & - & - \\
- & + & - & + & - & + & + \\
+ & - & - & + & - & + & - \\
- & - & - & + & - & + & + \\
\end{array}\right)
\]

We have the same contradiction as examples 1 and 4.

Our sixth example is figure 2.7

![Figure 2.7](image)

This represents \((-x - 1)(-2 + x - 2) = (-x - 1)(x - 4)\)

The related matrix is \(A = \left(\begin{array}{cccc}
- & - & + & - \\
- & - & + & - \\
\end{array}\right)\)

Again we have the same contradiction as examples 1, 4 and 5.

Our seventh example is figure 2.8.

![Figure 2.8](image)
This represents \((x + 1)(x - 4)\) The related matrix is
\[
A = \begin{pmatrix}
+ & - & - & - & - & + \\
+ & - & - & - & - & +
\end{pmatrix}
\]

When we check, \(A_{22}, A_{23}, A_{24}, A_{25}, A_{26},\) and \(A_{27}\), all follow the conditions for multiplying sign numbers. For example, \(A_{23} = A_{21} \cdot A_{13} = (+)(-) = -\) and \(A_{23}\) is negative. We have our second candidate.

Our last example is Figure 2.9

![Figure 2.9](image)

This represents \(x(2x)\) A black rectangle is multiplied by a white rectangle, is this a black square? This example has even stranger problems.

It is clear we need another condition.

Condition 2: the conditions for multiplying sign numbers must be followed.

We then have two possible answers, example 2 (figure 2.3) and example 7 (figure 2.8)

Looking more closely at example 2 (figure 2.3), we see that we used one \(x^2\), four negative \(x\)’s, one positive \(x\) and four negative ones, that is \(x^2 - 4x + x - 4\), which is factoring \(x^2 - 3x - 4\) by grouping. Also, in example 2, the top is \(x - 4\), the bottom is \(x - 4\), the left side is \(x + 1\) and the right side is \(-x - 1\). Example 2 physically is a rectangle, but geometrically it is not a rectangle since the left side and the right side are not the same, i.e. \(x + 1 \neq -x - 1\). Looking more closely at example 7 (figure 2.8), we see that we used one \(x^2\), five negative \(x\)’s, two positive \(x\)’s and five negative one’s, that is \(x^2 - 5x + 2x - 5 + 1\). Also, in example 7 (figure 2.8), the top is \(x - 4\), the bottom is \(x - 4\), the left side is \(x + 1\), and the right side is \(x + 1\).

Example 7 is the only one that is a physical rectangle and geometric rectangle. We example 7 is the correct answer, so we add another condition

Condition 3: the physical rectangle must be a geometric rectangle.

Our answer is
Example seven has one positive $x^2$, five negative x’s, two positive x’s, five negative ones and one positive one, that is $x^2 - 5x + 2x - 5 + 1$. The top of the rectangle is $x - 4$ and the left side is $x + 1$. How does one use $x^2 - 5x + 2x - 5 + 1$ to factor?

First, we use the top side $x - 4$ in the factoring $x^2 - 3x - 4$

$= x^2 - 5x + 2x - 5 + 1$

We rewrite $x^2 - 5x$ as three terms where the first two terms have $x - 4$ as a factor

$= x(x - 4) - x + 2x - 5 + 1$

We rearrange $-x + 2x - 5 + 1$ into four terms $ax + b + cx + d$ where $x - 4$ is a factor of $ax + b$ and $x - 4$ is a factor of $cx + d$

$= x(x - 4) + 2x - 8 - x + 4$

$= x(x - 4) + 2(x - 4) - (x - 4)$

$= (x - 4)(x + 2 - 1)$

$= (x - 4)(x + 1)$.

Next, we use the left side $x + 1$ in the factoring
\[= x^2 + 2x - 5x - 5 + 1\]

We rewrite \(x^2 + 2x\) as three terms where the first two terms have \(x + 1\) as a factor

\[= x^2 + x + x - 5x - 5 + 1\]

We rearrange \(x - 5x - 5 + 1\) into four terms \(ax + b + cx + d\) where \(x + 1\) is a factor of \(ax + b\) and \(x + 1\) is a factor of \(cx + d\)

\[= x(x + 1) - 5x - 5 + x + 1\]

\[= x(x + 1) - 5(x + 1) + (x + 1)\]

\[= (x + 1)(x - 5 + 1)\]

\[= (x + 1)(x + 4)\,.
\]

Note that \(x - 5 + 1\) is what is on the top and bottom of the rectangle.

We include more examples that satisfy conditions 1-3.

Example 9

![Figure 2.10](image-url)
The tiles used are one positive $x^2$, five negative $x$’s, two positive $x$’s, five negative one’s and seven positive one’s that is $x^2 - 3x + 2$ The top side is $x - 2$ and the left side is $x - 1$

First, we use the top side $x - 2$ in the factoring

\[ x^2 - 3x + 2 \]

\[ = x^2 - 5x + 2x - 5 + 7 \]

We rewrite $x^2 - 5x$ as three terms where the first two terms have $x - 2$ as a factor

\[ = x^2 - 2x - 3x + 2x - 5 + 7 \]

We rearrange $-3x + 2x - 5 + 7$ into four terms $ax + b + cx + d$ where $x - 2$ is a factor of $ax + b$ and $x - 2$ is a factor of $cx + d$.

\[ = x(x - 2) + 2(x - 2) - 3x + 6 \]

\[ = x(x - 2) + 2(x - 2) - 3(x - 2) \]

\[ = (x - 2)(x + 2 - 3) \]

\[ = (x - 2)(x - 1) \]

Next, we use the left side $x - 1$ in the factoring

\[ = x^2 - 5x + 2x - 5 + 7 \]

We rewrite $x^2 - 5x$ as three terms where the first two terms can be factored by $x - 1$
\[ x^2 - x - 4x + 2x - 5 + 7 \]

We rearrange \(-4x + 2x - 5 + 7\) into four terms \(ax + b + cx + d\) where \(x - 1\) is a factor of \(ax + b\) and \(x - 1\) is a factor of \(cx + d\).

\[ = x(x - 1) + 2x - 2 - 4x + 4 \]
\[ = x(x - 1) + 2(x - 1) - 4(x - 4) \]
\[ (x - 4 + 2)(x - 1) \]
\[ =(x-1) (x-2). \]

Note that \(x - 4 + 2\) is what is on the top and bottom of the rectangle

Example 10

![Figure 2.11](http://www.hostos.cuny.edu/mtrj/)

The tiles used are one positive \(x^2\), six negative \(x\)’s, three positive \(x\)’s, twelve negative one’s and two one’s. That is \(x^2−6x+3x−12+2\).

\[ x^2 -3x+10 \]
First, we use the top side $x - 5$ in the factoring

$$= x^2 - 6x + 3x - 12 + 2$$

We rewrite the first two terms ($x^2 - 6x$) as three terms where the first two terms can be factored by $x - 5$

$$= x(x - 5) - x + 3x - 12 + 2$$

We rearrange $-x + 3x - 12 + 2$ into four terms $ax + b + cx + d$ where $x - 5$ is a factor of $ax + b$ and $x - 5$ is a factor of $cx + d$.

$$= x(x - 5) + 3x - 15 - x + 5$$

$$= x(x - 5) + 3(x - 5) - (x - 5)$$

$$= (x - 5)(x + 3 - 1)$$

$$= (x - 5)(x + 2)$$

Next, we use the left side $x + 2$ in the factoring

$$x^2 - 3x - 10$$

$$= x^2 + 3x - 6x - 12 + 2$$

We rewrite the first two terms $x^2 + 3x$ as three terms where the first two terms can be factored by $x + 2$

$$= x^2 + 2x + x - 6x - 12 + 2$$
We rearrange $x - 6x - 12 + 2$ into four terms $ax + b + cx + d$ where $x + 2$ is a factor of $ax + b$ and $x + 2$ is a factor of $cx + d$.

$$= x(x + 2) - 6x - 12 + x + 2$$

$$= x(x + 2) - 6(x + 2) + x + 2$$

$$= (x + 2)(x - 6 + 1)$$

$$= (x + 2)(x - 5)$$

Note that $x - 6 + 1$ is what is on the top and bottom of the rectangle

The next two examples we leave it to the reader to verify the tiles.

Example 11: $x^2 - 3x - 18$

The tiles used are: One positive $x^2$, seven negative x’s, four positive x’s, twenty-one negative one’s, and three one’s. That is $x^2 - 7x + 4x - 21 + 3$.

First, we use the top side $x - 5$ in the factoring

$$x^2 - 3x - 18$$

$$= x^2 - 7x + 4x - 21 + 3$$

We rewrite the first two terms $x^2 - 7x$ as three terms where the first two terms can be factored by $x - 6$

$$= x^2 - 6x - x + 4x - 21 + 3$$
We rearrange $-x + 4x - 21 + 3$ into four terms $ax + b + cx + d$ where $x - 6$ is a factor of $ax + b$ and $x - 6$ is a factor of $cx + d$.

$$= x(x - 6) + 4x - 24 - x + 6$$

$$= x(x - 6) + 4(x - 6) - x + 6$$

$$= x(x - 6) + 4(x - 6) - (x - 6)$$

$$= (x - 6)(x + 4 - 1)$$

$$= (x - 6)(x + 3)$$

Next, we use left side $x + 3$ in the factoring

$$x^2 - 3x - 18$$

$$= x^2 + 4x - 7x - 21 + 3$$

We rewrite the first two terms $x^2 + 4x$ as three terms where the first two terms can be factored by $x + 3$

$$= x(x + 3) + x - 7x - 21 + 3$$

We rearrange $x - 7x - 21 + 3$ into four terms $ax + b + cx + d$ where $x + 3$ is a factor of $ax + b$ and $x + 3$ is a factor of $cx + d$.

$$= x(x + 3) - 7x - 21 + x + 3$$

$$= x(x + 3) - 7(x + 3) + x + 3$$
\[(x + 3) (x - 7 + 1)\]

\[(x + 3) (x - 6)\]

Note that \(x - 7 + 1\) is what is on the top and bottom of the rectangle.

Example 12: \(6x^2 - x - 15\)

The tiles used are: Six positive \(x^2\), twelve negative \(x\)'s, eleven positive \(x\)'s, eighteen negative ones and three one's.

\[6x^2 + 11x - 12x - 18 + 3\]

One way is to rewrite the first two terms \(6x^2 + 11x\) as three terms where the first two terms can be factored by \(2x + 3\), were \(2x + 3\) is the top side.

\[= 3x(2x + 3) - 6(2x + 3) + 2x + 3\]

\[= (2x + 3)(3x - 6 + 1)\]

\[= (2x + 3)(3x - 5)\]

We again look at \(6x^2 - x - 15\)

\[= 6x^2 - 12x + 11x - 18 + 3\]

The other way is to rewrite the first two terms \(6x^2 - 12x\) as three terms where the first two terms can be factored by \(3x - 5\), where \(3x - 5\) is the left side.

\[= 6x^2 - 10x - 2x + 11x - 18 + 3\]
\[= 2x(3x - 5) + 3(3x - 5)\]

\[= (3x-5) (2x +3).\]

We look at \(x^2 + ax + c\). If \(a\) and \(b\) are positive integers, the rectangle will follow conditions 1-3. If \(a\) and \(b\) are negative integers, the rectangle will not follow condition 3, since the left side ≠ rightside, or top ≠ bottom. If \(a\) is negative and \(b\) are positive integers, the rectangle will not follow condition 3, since the left side ≠ rightside and top ≠ bottom.

Section 3 Comparison of the Two Methods.
B.C. thinks the difference shows that there are two different ways of mathematical thinking. T.B.’s approach shows the beauty and elegance of the mathematical problem solving by reasoning, the problem being how many rectangles are needed to factorize the trinomial expression, while B.C. (a physicist) presents a direct approach to the stated problem with the help of Aha! Moment insight. The direct approach yielded a different type of tile model solutions presented than those presented by T.B. in the second section. In order to convey the correct tile model for a given trinomial, B.C. utilized the third dimension in the representation of \(x - 1\) by putting tiles on top of tiles. In order to convey the correct tile model for the given trinomial, T.B. utilized the second dimension in the representation of \(x - 1\) by not putting tiles on top of tiles but by adding them in the same 2-dimensional plane. This “spatial” difference between the two reflects the difference between \(x - 1\) and \(x + (-1)\). Whereas the latter algebraic expression symbolizes addition of the negative on the plane, \(x - 1\) symbolizes taking away 1 unit from the side \(x\).

T.B. also thinks the difference is how each view the problem. He sees B.C. as a two-layer problem and his own as a one-layer problem. Both B.C. and T.B. used \(x^2 - 3x - 4\) and the algebra tiles to form their rectangles. Both used the number of squares, positive x’s, negative x’s, positive one’s and negative one’s from their rectangle to factor \(x^2 - 3x - 4\). The difference is in how they made their rectangles. Figure 3.1 is B.C. two-layer version.
His answer is the lower level that is not covered. This rectangle is both a physical and geometric rectangle. He used “four terms” $x^2 - 4x + x - 4$ and this is factoring by grouping. T.B. saw this same problem as a one-layer problem. To get his rectangle to be both a physical and geometric rectangle, he needed more tiles than B.C.

Figure 3.1 is T.B. one-layer version

From his rectangle he used “five terms” $x^2 - 5x + 2x - 5 + 1$. He then developed a way to factor $x^2 - 5x + 2x - 5 + 1$ which is unusual. What is unusual is one factors that he gets is: $x - 5 + 1$, which is the top and the bottom rows of his rectangle. He used “five terms” and changed them into “six terms”. With the “six terms”, he used the factoring by grouping twice. Comparing both methods with the available resources on line (e.g., teachers.desmos.com) we find out no two layer approach and no new approach to factorization of $ax + b + cx + d$ presented in the text.

**References**
