Editorial from Małgorzata Marciniak, a Managing Editor,

In September 2017, I began my duties as a new Managing Editor of the Mathematics Teaching-Research Journal and I have to admit that that moment entirely changed my perspective on writing and reading. Since that time, the process of writing became a permanent companion of mine. Now, writing articles, reviews, and applying revisions carries the second momentum for writing another article, which becomes a cycle rather than a linear process that I was used to.

The journal changed me, and I hope that the journal is changing as well. As one can see, MTRJ now has the ISSN number, new guidelines for authors and new ethics statement. We began accepting manuscripts in LaTeX format and added page numbering to the journal content. Starting with Volume 9, N 3-4, we will have a new logo and new appearance of the articles. The new logo is inspired by the flow of arrows on the Mobius band, which appear to flow from one point to opposite directions, but they eventually converge. Just like teaching and research seem to part as different aspects of academic work but they eventually arrive to the same point of sharing research skills with our own students.

This volume contains four articles that analyze teaching tools and theories of learning.

"Pilot study of the effect of the use of cultural materials and women's stories on the academic achievement of senior secondary students in geometry in Abuja, Nigeria" by Agwu Nkechi, *et al* contains a study of the influence of implementation of ethnical and female stories on the accomplishments of high school students. The study concluded that the use of cultural materials and women's stories in teaching geometry improve the academic achievement of students in general, irrespective of their sex.

"The `act of creation' of Koestler & theories of learning in math education research" by William Baker and Bronisław Czarnocha discusses Koestler's bisociation as a type of reflective abstraction that provides a mechanism of concept development to transition the solver through the Piaget & Garcia Triad. The article contains an example from the classroom that visualizes the Koestler's theory of creativity.

"Brief creative assignments in undergraduate mathematics courses: Calculus 3 and Linear Algebra" by Małgorzata Marciniak. This article is a note from classroom experiments with creative assignments. The theory behind it was inspired by Graham Wallace's classic from 1926 that contains the description of stages of creative thoughts. Trying to align this concept in a math classroom one faces multiple challenges with the most difficult one: how to observe the stages of creative thought that students are going through.

"Least Squares Estimation" instructional design based upon APOS theory: Laying Mathematical Representation and Transformation Bridge" by Yangchun Xie describes the theory behind the introduction of the linear regression to high school students in China. She presents her experience with the four stages of APOS (Action-Process-Object-Schema) and review literature related to the APOS theory. It is fascinating to observe the Chinese take upon that American contemporary theory of conceptual development in mathematics. In her unusual conceptual framework, she goes beyond APOS theory in seeing the process of understanding the Least Squares method as the process from the randomness of events to the certainty of the functional relationship expressed by the straight line obtained with the help of the Least Squares method.

Needless to say, I am thrilled to become a Managing Editor of the Mathematics Teaching-Research Journal. Now, when Volume 9, N 3-4, is ready for release, I can see the process of creation from another perspective.

Małgorzata Marciniak

## Contents

# Pilot study on the effect of the use of cultural materials and women's stories on the academic achievement of senior secondary school students in geometry in Abuja, Nigeria

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***Abstract:*** *This pilot study was to determine the effect of the use of cultural materials on the achievement of senior secondary school students in geometry. The one group pretest post-test research design was adopted for the study. A purposive sampling procedure was used to sample 25 senior secondary school students. The main instrument for the collection of data was a Researcher questionnaire titled "Geometry Test for SS1 students." The logical validity of the instrument was 87%. Also, its reliability was determined using split- half method and the reliability was 80%. The treatment lasted three hours for a duration of two days. The data were analyzed using mean, standard deviation, paired sample t-test and independent t-test. The results of the study revealed that there was a significant difference in the pre- test mean and post-test mean geometry scores of students taught geometry using cultural materials and women's stories. Also, although the mean post test score of male students in geometry is higher than that of the female students, there is no significant difference in the mean post-test mean achievement scores in geometry between the male and the female students taught with the use of cultural materials and women's stories. The study concluded that the use of cultural materials and women's stories in teaching geometry improves the academic achievement of students in general, irrespective of their sex. It was therefore recommended that all mathematics teachers should adopt the use of cultural materials and women stories in the teaching of geometry in secondary schools. Also, all* *mathematics teachers should be made to undergo a capacity building workshop on the use of cultural materials and women stories in the teaching of geometry in secondary schools.*

Keywords: Cultural materials, Women's stories, Geometry, Academic Achievement, Senior Secondary School

**Background of the study**

The importance of Science and Technology in the world today cannot be overemphasized. Science and technology have become an integral part of the World's culture. Hence, the study of mathematics is made compulsory at the Primary and Secondary levels of education in Nigeria (Federal Republic of Nigeria (FRN, 2013) and many other countries of the world. The difference in the level of development between the developed and the underdeveloped countries can be traced to the difference in the teaching and learning of Mathematics in those countries. This is due to the fact that Mathematics is the bedrock of Science and Technology which are the core ingredients for modern development (Ukeje, 2002). Every day human activities are driven by Science and Technology; and undoubtedly, a sound Science, Technology, Engineering, Arts and Mathematics (STEAM) education is the key to good health, development of industries, poverty alleviation, promotion of peace, conservation of environment, and good life for all and improved economic growth and development (Anwukah, 2017). The foundation for a STEAM career is laid early in life, but scientists and engineers are made in colleges and universities. (American Association of University Women (AAUW), 2016). It is an undisputable fact that many students in Nigerian Secondary Schools encounter multiple problems with the study of Mathematics. The trend in the academic achievement of secondary school students in Mathematics in the last two decades has become a source of concern to all stakeholders in the education sector including parents, students, school administrators and the general public. Students have not been performing well in Mathematics in most examinations. It has been asserted that academic failure is not only frustrating to the students and the parents, but its effects are equally grave on the society in terms of dearth of manpower in all spheres of the economy and polity (Aremu, 2000; Morakinyo, 2003).

This problem has two main dimensions. While students on one hand have difficulties in understanding the topics taught, teachers on the other hand equally have difficulties in achieving effective teaching in our schools. Another dimension of the problem is that in most societies in Nigeria, the girl child education is not in the front burner. Women are left behind in most professions. In particular, the number of women in STEAM education is low when compared with the male counterparts. The striking difference between the number of men and women in Science, Technology, and Mathematics has often been considered as evidence of biologically driven gender differences in abilities and interests. The classical foundation of this is that men "naturally" excel in mathematically demanding disciplines; whereas women "naturally" excel in fields using language skills (AAUW, 2016).

Literature on the achievement in mathematics with respect to gender has remained inconclusive. Some researchers have found that male students perform better than female students in Mathematics (Atovigba, Vershima, O'kwu & Ijenkeli, 2012; Ali, Bhagawati & Sarmah, 2014). Also, Timayi, Ibrahim and Sirajo (2016) in their study found that there was difference in the mean and standard deviation scores of male and female students in favor of the male students in geometry test, but the observed difference was not statistically significant with regard to achievement and gender interaction. Some other researchers (Abubakar & Adeboyega, 2012) reported that gender has no significant effect on the achievement of students. But some researchers (Linderberg, Hyde, Petersen & Lin (2010) reported that gender differentials among males and females is converging; hence, they perform similarly. Moreover, the effect of gender requires further research as it may affect achievement especially in Mathematics.

The glaring problems faced by students in the study of Mathematics is evidenced by the poor performances of students in internal and external examinations in Mathematics. For instance, for the years between 2002 and 2014 Nigeria did not record up to 60% credit pass in Mathematics in the West African Senior Secondary School Certificate Examination (West African Examinations Council (WAEC), 2002, 2007, 2011, 2012a, 2012b, 2013, 2014). Also, the head of WAEC National office, Adenikpekum (2017), reported that 59.22% of Nigerian candidates had a minimum of five credits in senior secondary school subjects including Mathematics and English Language. Some of the challenges affecting the teaching and learning of Mathematics in Nigerian schools include: lack of textbooks on the part of students, inadequate Mathematics instructional materials, insufficient mathematics teachers, poor student background, and poor condition of learning environment (Federal Ministry of Education (FME), 2012). It has been observed that among the factors that influence achievement of learners of school Mathematics, teachers' effectiveness as measured through the acquisition and use of good instructional skills and methodologies appear very prominent (Max, 1988). Studies have shown that high-quality teaching can make a significant difference in students learning and high-quality teaching requires a high-quality workforce (Egbo, 2011; Anwuka, 2017). It is true that no educational system can rise above its teachers because no nation can rise above the standards of its schools (Federal Republic of Nigeria (FRN), 2013). Thus, education is the key that unlocks the door to modernization, but it is the teacher who holds the key to the door.

Mathematics is considered by many to be abstract. Most teachers teach Mathematics abstractly because most schools do not have Mathematics laboratories. For instance, research has shown that many schools have no Mathematics laboratory, and Mathematics teachers indicated that using Mathematics laboratory to teach Mathematics makes abstract topics to be more concrete to the students (Sunday, Akanmu, Salman & Fajemidagba, 2016). There is, therefore, a dare need for special intervention in schools in the area of retraining of teachers and use of innovative teaching materials for teaching Mathematics in schools. For instance, the National Mathematical Centre worked on some schools in some states under the Mathematics Improvement Project (MIP). After the intervention by the Centre, the results of the schools improved tremendously. In Kogi State, the percentage credit pass in Mathematics of G.S.S Icheke-Ogane rose from 7.69% in 2010 to 55.56% in 2012, and that of St Peters' College Idah rose from 33.02% in 2010 to 70.86% in 2012. Also, in Kaduna State, the percentage credit pass in Mathematics of GSS Makarfi rose from 29.9% in 2013 to 99.2% in 2015 after the intervention by the Centre under the Mathematics Improvement Project (National Mathematical Centre, 2015)

One of the new and innovative ways of teaching Mathematics and the Sciences in schools is the use of Cultural materials and women's stories. This is designed for capacity building in STEAM related fields and for fostering entrepreneurship and innovations. This is part of Ethno-mathematics. Ethno-mathematics is defined as "the Mathematics of identifiable cultural group derived from quantitative and qualitative practices like counting, weighing, sorting, measuring and comparing" (D'Ambrosio, 1985). Enukoha (1995) defined Ethno-mathematics as "a discipline interested in studying Mathematics and Mathematics education in the cultural milieu of the learner.” Aprebo (2016) recommended that the use of teaching aids in our environment for Mathematics teaching. He emphasized that the use of African objects as examples in the teaching of Mathematics makes the learner to identify the application of Mathematics in any subject that he/she handles and make him/her to see the Mathematical composition in any object he/she studies. Also, Ugwuanyi (2014) opined that the use of instructional materials in Mathematics reduces to a large extent the abstract nature of many Mathematical concepts. To sustain students’ interest in Mathematics, the teaching of the subject should be practical, exploratory and experimental which could be carried out in the Mathematics laboratory (Salman, 2002). The use of Ethno-mathematics has been found to improve the achievement and interest of students in geometry and measurement (Kurumeh, 2004). Also, Enukoha (1995) emphasized that there is no culture in which the rudiments of Mathematics such as counting, measuring, locating, designing, reasoning, exploring, playing games, adding, subtracting, multiplying, dividing, and some other cognitive activities are not carried out There is no corner of any society that there are no materials and activities that are not Mathematical in nature. These can be found in the shapes of our traditional houses and roofs, local pots, bracelets, local woven baskets, furniture, buying and selling, etc. Therefore, the use of cultural materials and stories could be used to improve the effectiveness of teaching and learning of Mathematics in schools.

In Nigeria today, this innovative approach to teaching is not yet utilized by many teachers while in many advanced countries this approach is used for teaching all subjects at all levels. The use of culture and women’s stories are used to concretize educational concepts, arouse and sustain the interest of learners, and to foster entrepreneurship and innovation among learners. With this innovative approach, the achievement of students in the study of STEAM could be improved upon. Unfortunately, many teachers in Nigeria are not employing this approach of teaching.

Now the problem of the teacher in terms of quantity and quality has been fingered as the one of the most important factors affecting the performance of students in Mathematics. In particular, the approach to the teaching of the Mathematical Sciences is an important factor that determines the achievement of students in the Mathematical Sciences. This approach of using cultural materials and women's stories has been used sparingly in Nigeria. The problems of Mathematics students resulting from the approach of teachers deserve appropriate attention.

**Statement of the Problem**

Mathematics knowledge is applied to all aspects of human endeavors. Hence, the study of Mathematics is made compulsory at the Primary and Secondary levels of education in Nigeria and many other countries of the world. Unfortunately, the performance of students in Mathematics over the years has remained unsatisfactory. This has prevented many students from studying scientific and technological courses in tertiary institutions and this consequently negates the economic and technological strides of the Country. Mathematics is the main ingredient for the economic and technological development of any nation. One of the major problems facing the teaching and learning of Mathematics is the lack of instructional materials to concretize the mathematical concepts and sustain the interest of learners in the study of the subject. The use of cultural materials and stories has been proposed to be used in the teaching of Mathematics in schools as a means for improving the academic achievement of students and to sustain their interest in the teaching of the subject. But not much research has been done on this in Nigeria. Hence, the need for this study.

**Purpose of the Study**

The purpose of this study was to determine the effect of the use of cultural materials and women's stories on the academic achievement of senior secondary school students in Abuja. Specifically, the study has the following objectives to:

I. investigate the effect of the use of cultural materials and women's stories on the mean academic achievement of senior secondary school students in Geometry.
II. determine the effect of the use of cultural materials and women's stories on the mean academic achievement of male senior secondary school students in Geometry.
III. determine the effect of the use of cultural materials and women's stories on the mean academic achievement of female senior secondary school students in Geometry.

IV. investigate the difference in the mean post test scores in geometry between male and female students taught with the use of cultural materials and women's stories

**Research Questions**

To guide the study, the following research questions were formulated:

i. how would the use of cultural materials and women's stories affect the mean academic achievement of senior secondary school students in Geometry?
ii. what is the effect of the use of cultural materials and women's stories on the mean academic achievement of male senior secondary school students in Geometry?
iii. how would the use of cultural materials and women's stories affect the mean academic achievement of female senior secondary school students in Geometry?
iv. what is the difference in the mean post-test achievement score in Geometry between male and female students taught with the use of cultural materials and women's stories?

**Statement of the Hypotheses**

$Ho_1$ : There is no significant difference in the pre-test mean and post-test mean academic achievement of senior secondary school students in Geometry taught using cultural materials and women's stories.

$Ho_2$ : There is no significant difference in the pre-test mean and post-test mean academic achievement of male senior secondary school students in Geometry taught using cultural materials and women's stories.

$Ho_3$: There is no significant difference in the pre-test mean and post-test mean academic achievement of senior secondary school students in Geometry taught using cultural materials and women's stories.

$Ho_4$ : There is no significant difference in the post test mean achievement scores in geometry between the male and female students taught with the use of cultural materials and women's stories

**Methodology**

The research design adopted for the study was the One-Group Pretest Posttest Design. The design was adopted because it was not possible to have a control group for the study at that point in time. The population consisted of all the students of the International Model Science Academy which is the Demonstration School for the National Mathematical Centre, Abuja. However, Convenience sampling was used to sample an intact class of 25 Senior Secondary one students consisting of 12 males and 13 females. It was convenient for the researchers to reach the students since the school was very accessible to the researchers. Being a model Academy, the students are from over six States and Federal Capital Territory (FCT) of Nigeria. Three Mathematics teachers also participated in the project.

The instrument for the study was a Researcher made Questionnaire titled "Geometry Test for SS1 students." The questionnaire was validated by four experts in Mathematics Education and Mathematics from National Mathematical Centre, Abuja using face validation and table of specification to ensure the content validity. The logical validity was 0.87. Also, the reliability was determined using split half method and the reliability was 0.80. The instrument consisted of 12 short subjective questions. The topics covered were circumference, area, annulus and other properties of circle; volume and capacity of solid shapes; and related financial Mathematics. There were pre-test and post-test. The post-test was a reshuffled pre-test. The pre- test was administered at the beginning of the workshop and the duration was thirty minutes. The questionnaires were collected from the participants after the pre-test and the participants were not informed that there would be a post test. The Researchers had discussion, demonstration and practical lessons with the participants for 3 hours per day for a period of two days. The lessons covered, the teaching of the identified topics using cultural materials such as bracelets, local woven baskets, local pots, etc. At the end of the program, a post-test was given to the participants which lasted for thirty minutes. Both the pre-test and the post-test questionnaires were marked and recorded. The data were analyzed using mean, standard deviation, paired sample t-test and independent t-test.

**Presentation of Results**

**Research Question 1($RQ_1$)**

How would the use of cultural materials and women's stories affect the mean academic achievement of senior secondary school students in Geometry?

**Table 1: Pre-Test, Post-Test Mean and Standard Deviations Geometry Scores of Students Taught with the use of Cultural Materials and Women's stories.**

| | | Mean | N | Std. Deviation | Std.Error Mean |
|---|---|---|---|---|---|
| Pair 1 | Pre | 31.7200 | 25 | 17.21801 | 3.44360 |
| | Post | 62.3600 | 25 | 21.29335 | 4.25867 |

Table 1 shows the Pre-Test, Post-Test Mean and Standard Deviation of Students taught with the use of cultural materials and women's stories. The pre-test mean score was 31.7200 with standard deviation of 17.21 while the post-test mean score was 62.3600 showing increase mean of 31.4400. This implies that students taught with the use of cultural materials and women's stories had improvements in students' mean academic achievement in Geometry.

**Null Hypothesis $H_{01}$**

There is no significant difference in the pre-test mean and post-test mean academic achievement of senior secondary school students in Geometry taught using cultural materials and women's stories.

**Table 2: Summary of Paired Sample t-Test of the Pre-Test and Post-Test Geometry Scores of Students Taught with the Use of Cultural Materials and Women's stories.**

| | | Paired Differences | | | | | t | df | Sig.(2-tailed) |
|---|---|---|---|---|---|---|---|---|---|
| | | Mean | Std. Deviation | Std.Error Mean | 95% Confidence Interval of the Difference | | | | |
| | | | | | Lower | Upper | | | |
| Pair 1 | Pre-Post | -30.64000 | 19.76714 | 3.95343 | -38.79948 | -22.48052 | -7.750 | 24 | .000 |

Table 2 shows the paired sample t-test of the pre-test and post-test mathematics Geometry Scores of Students taught with the use of cultural materials and women's stories. From the table, significant 2-tailed value (P-value) is 0.000. Since the value of 0.000 is less than 0.05, therefore the null hypothesis is rejected. This implies that there was a significant difference in the pre-test mean Geometry Scores of Students taught with the use of cultural materials and women's stories and their post-test mean Geometry scores. Thus, the students taught using cultural materials and women's stories significantly improved their mean academic achievement in Geometry.

**Research Question 2 ($RQ_2$)**

What is the effect of the use of cultural materials and women's stories on the mean academic achievement of male senior secondary school students in Geometry?

**Table 3: Pre-Test, Post-Test Mean and Standard Deviations Geometry Scores of Male Students Taught with the Use of Cultural Materials and Women's stories**

| | | Mean | N | Std. Deviation | Std. Error Mean |
|---|---|---|---|---|---|
| Pair 1 | Pre | 35.7500 | 12 | 16.81517 | 4.85412 |
| | Post | 68.7500 | 12 | 15.29186 | 4.41438 |

Table 3 shows the Pre-Test, Post-Test Mean and Standard Deviations Geometry Scores of Male Students taught with the use of cultural materials and women's stories. The pre-test mean score was 35.7500 with standard deviation of 16.815 while the post-test mean score was 68.75 with standard deviation of 15.291 showing in increase mean of 33. This implies that the male Students taught with the use of cultural materials and women's stories had improvements in their mean academic achievement in geometry.

**Hypothesis $H_{02}$**

There is no significant difference in the pre-test mean and post-test mean academic achievement of male senior secondary school students in Geometry taught using cultural materials and women stories.

**Table 4: Summary of Paired Sample t-Test of the Pre-Test and Post-Test Geometry Scores of Male Senior Secondary School Students in Geometry Taught Using Cultural Materials and Women's stories.**

| | | Paired Differences | | | | | t | df | Sig. (2-tailed) |
|---|---|---|---|---|---|---|---|---|---|
| | | Mean | Std. Deviation | Std.Error Mean | 95% Confidence Interval of the Difference | | | | |
| | | | | | Lower | Upper | | | |
| Pair 1 | Pre-Post | -33.0000 | 17.16233 | 4.95434 | -43.90442 | -22.09558 | -6.661 | 11 | .000 |

Table 4 shows the paired sample t-test of the pre-test and post-test geometry scores of male senior secondary school students in Geometry taught using cultural materials and women's stories. From the table, significant 2-tailed value (P-value) is 0.000. Since the value of 0.000 is less than 0.05, therefore the null hypothesis is rejected. This implies that there was a significant difference in the pre- test mean geometry scores of male senior secondary school students in Geometry taught using cultural materials and women's stories and their post-test mean Geometry scores. Thus, the male senior secondary school students taught using cultural materials and women's stories significantly improved their mean academic achievement in Geometry.

**Research Question 3 ($RQ_3$)**

How would the use of cultural materials and women's stories affect the mean academic achievement of female senior secondary school students in Geometry?

**Table 5: Pre-Test, Post-Test Mean and Standard Deviations Geometry Scores of Female Students Taught with the Use of Cultural Materials and Women's stories**

| | | Mean | N | Std. Deviation | Std. Error Mean |
|---|---|---|---|---|---|
| Pair 1 | Pre | 28.0000 | 13 | 17.39253 | 4.82382 |
| | Post | 56.4615 | 13 | 24.77773 | 6.87211 |

Table 5 shows the pre-test, post-test mean and standard deviations Geometry scores of female students taught with the use of cultural materials and women's stories. The pre-test mean score was 28.00 with standard deviation of 17.39 while the post-test mean score was 56.46 with standard deviation of 24.78 showing in increase mean of 28.46 This implies that the female Students taught with the use of cultural materials and women's stories had improvements in their mean academic achievement in Geometry.

**Hypothesis $H_{03}$**

There is no significant difference in the pre-test mean and post-test mean academic achievement of female senior secondary school students in Geometry taught using cultural materials and women's stories.

**Table 6: Summary of Paired Sample t-Test of the Pre-Test and Post-Test Geometry Scores of Male Senior Secondary School Students in Geometry Taught Using Cultural Materials and Women's stories.**

| | | Paired Differences | | | | | t | df | Sig. (2-tailed) |
|---|---|---|---|---|---|---|---|---|---|
| | | Mean | Std. Deviation | Std.Error Mean | 95% Confidence Interval of the Difference | | | | |
| | | | | | Lower | Upper | | | |
| Pair 1 | pre - post | -28.46154 | 22.37787 | 6.20651 | -41.98435 | -14.93872 | -4.586 | 12 | .001 |

Table 4 shows the paired sample t-test of the pre- test and post-test-geometry scores of female senior secondary school students in Geometry taught using cultural materials and women's stories. From the table, significant 2-tailed value (P-value) is 0.001. Since the value of 0.001 is less than 0.05, therefore the null hypothesis is rejected. This implies that there was a significant difference in the pre- test mean geometry scores of female senior secondary school students in Geometry taught using cultural materials and women's stories and their post -test mean geometry scores. Thus, the female senior secondary school students taught using cultural materials and women's stories significantly improved their mean academic achievement in Geometry.

**Research Question 4 ($RQ_4$)**

What is the difference in the mean post-test mean achievement score in Geometry between male and female students taught with the use of cultural materials and women's stories

**Table 7: Post- Test Mean and Standard Deviations Geometry Scores of Male and Female Students Taught with the Use of Cultural Materials and Women's stories**

| | Group | N | Mean | Std. Deviation | Std. Error Mean |
|---|---|---|---|---|---|
| Score | Male | 12 | 68.7500 | 15.29186 | 4.41438 |
| | Female | 13 | 56.4615 | 24.77773 | 6.87211 |

Table 7 shows the post-test mean and standard deviations Geometry scores of male and female students taught with the use of cultural materials and women's stories. The table shows that male students had mean post-test score of 68.75 with standard deviation of 15.29 while the female students had mean post-test score of 56.46 with standard deviation of 24.78. This shows that the male students perform better than female students in the Geometry test.

**Hypothesis $H_{04}$**

There is no significant difference in the post-test mean achievement scores in Geometry between the male and female students taught with the use of cultural materials and women's stories

Table 8: **Summary of Independent Sample t-test of the Post-test Geometry Scores of Male and Female Senior Secondary School Students Taught Using Cultural Materials And Women's stories**.

| Variable | N | Mean | SD | S Error | Mean difference | df | t-value | p-value | Decision |
|---|---|---|---|---|---|---|---|---|---|
| **Male** | 12 | 68.75 | 15.29 | 4.41 | 12.29 | 23 | 1.48 | 0.153 | Ho Not rejected |
| **Female** | 13 | 56.46 | 24.78 | 6.87 | | | | | |

Table 8 shows the summary of independent sample t-test of the post-test Geometry scores of male and female Senior Secondary School Students taught using cultural materials and women's stories. The P-value is 0.153. Since, 0.153 > 0.05, then the null is upheld. This implies that there is no significant difference in the post- test mean achievement scores in Geometry between the male and female students taught with the use of cultural materials and women's stories.

**Discussion of Results**

The results of the study showed that the use of cultural materials and women's stories improve the academic achievement of both male and female students in Geometry. This study supports the finding by Kurumeh (2004) that the use of Ethno mathematics improves the achievement and interest of students in Geometry and Measurement (Kurumeh, 2004). It also lends support to Aprebo's (2016) recommendation that the use of teaching aids in our environment for Mathematics teaching and the use of African objects as examples in the teaching of Mathematics make the learners bring out the applications in any subject that he/she handles and make him/her to see the mathematical composition in any object he/she studies Also, it supports Ugwuanyi (2014) who opined that the use of instructional materials in Mathematics reduces to a large extent the abstract nature of many mathematical concepts. When the Mathematics topics are made less abstract, the understanding and retention of concepts are improved upon and this leads to higher academic achievement.

The result of the study also shows the mean score of students in Geometry is higher than that of the female students but there is no significant difference in the mean post-test mean achievement scores in Geometry between the male and female students taught with the use of cultural materials and women's stories. This finding supports some researchers (Atovigba, Vershima, O'kwu & Ijenkeli, 2012; Ali, Bhagawati & Sarmah, 2014) who found that male students perform better than female students in Mathematic, and Timayi, Ibrahim and Sirajo (2016) who in their study found that there was difference in the mean and standard deviation scores of male and female students in favour of the males students in geometry test but the observed difference was not statistically significant with regard to achievement and gender interaction. However, this result does not support some other researchers Linderberg, Hyde, Perersen & Lin (2010) who reported that gender differentials among males and females is converging; hence, they perform similarly.

**Conclusion**

From the results of the study, the use of cultural materials and women's stories in teaching Geometry improves the academic achievement of students in general, irrespective of the gender of the students. Also, although the post-test mean score of students in Geometry is higher than that of the female students, there is no significant difference in the post-test mean achievement scores in Geometry between the male and female students taught with the use of cultural materials and women's stories.

**Recommendations:**

It is hereby recommended that:

a) all Mathematics teachers should adopt the use of cultural materials and women's stories in the teaching of Geometry in secondary schools.
b) all Mathematics teachers should be made to undergo a capacity building workshop on the use of cultural materials and women's stories in the teaching of geometry in secondary schools.

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# The "act of creation" of Koestler & theories of learning in math education research.

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***Abstract:*** *The presentation explores the relationships between Koestler's theory of the Act of Creation and learning theories used in mathematics education, in the context of the call by Prabhu & Czarnocha (2014) To provide a suitable background Koestler's understanding of creativity as a transformation from habit to originality is integrated with Vygotsky's understanding of concept development within the common learning environment of problem solving. The presentation provides an instructional example of bridging Koestler's insistence on the individual, "un-tutored" nature of bisociation with Vygotsky's concept of ZPD within properly structured social learning environment. In this instructional sequence Koestler's bisociation is seen as a type of reflective abstraction that provides a mechanism of concept development to transition the solver through the Piaget & Garcia Triad (1989).*

Keywords: bisociation, creativity,

## Bisociation as the definition of creativity in mathematics education

Prabhu & Czarnocha (2014) call for the adoption of Koestler bisociation as the definition of creativity in mathematics education in response to the conviction of the leaders of the field that there is no single, authoritative perspective or definition of creativity (Mann, 2005; Sriraman, 2005; Leikin, 2009, Kattou et al, 2011). The theory of the Act of Creation (1964) by Arthur Koestler, defines "bisociation" as the "creative leap [of insight], which connects the previously unconnected frames of reference and makes us experience reality at several planes at once" (p.45) an Aha moment.

In following words Koestler clarifies the experience [of] reality at several planes at once: "The pattern… is, the perceiving into situation or Idea, L, in two self-consistent but habitually incompatible frames of reference, M1 and M2. The event L, in which the two intersect, is made to vibrate simultaneously on two different wavelengths, as it were. While this unusual situation lasts, L is not merely linked to one associative context, but bisociated with two" (p.35).

Prabhu and Czarnocha (2014) state two reasons for their call:

1. Democratization of the research into mathematical creativity, which, in their opinion, is presently biased towards giftedness.

2. Separation of the definition of creativity from the concept of fluency since recent reports indicate that emphasis on fluency might negatively impact creativity itself, resulting in the lowering of originality as reported by Leikin, (2009).

An important component of Koestler theory, which makes it particularly useful for democratization in learning mathematics by all students is the relationship between creativity, originality and habit, in often quoted words of Koestler, "*Creativity is the defeat of habit by originality*." Stated in more positive and precise terms, bisociation can help transform habits that hamper learning of mathematics into original creative insights of our students (Prabhu & Czarnocha, 2014). It's helpful to know that our colleagues in Computer creativity branch of informatics have identified already three types of bisociation: bisociation by a common concept, bisociation within a concept map, and bisociation by the sub-structural similarity. Thus, the groundwork for a teaching-research investigation of bisociativity in our classrooms has been prepared (Berthold, 2012).

To ease the process of adoption of bisociativity into the mathematics education discourse we present here elements of a very rich interaction between Koestler's theory and different theories of learning utilized in mathematics education: Approaches of schema development based upon Piaget's work, and Vygotsky theory of concept development. This short expose will provide a unifying impetus for these diverse theories of learning based upon the relationship of bisociation with the theme of consciousness during reflection and abstraction within problem solving.

Koestler believed that creativity, specifically bisociation took place within a problem solving environment as a result of intense reflection and conscious deliberation. Vygotksy would affirm that concept development takes within a problem solving environment as a result of "reflective consciousness." Educators who follow the work of Piaget understand concept development and schema formation as the result of "reflective abstraction" within a problem solving environment.

**Koestler and Piaget**

Bisociation is the process, which can take place in the context of the construct called dialectical Triad of Piaget and Garcia formulated in the profound but little known book, Psychogenesis and the History of Science (Piaget & Garcia, 1989). This Triad of stages for concept formation was constructed on the basis of the comparative analysis of the development of physical and mathematical ideas in history of science on one hand, and the psychogenetic development of these concepts in a child, on the other. It is defined as the process of concept development through intra-operational, inter-operational and trans-operational stages.

"Intra-operational stages are characterized by intra-operational relations, which manifest themselves in forms that can be isolated"

inter-operational stage is "characterized by correspondences and transformations among the forms that can be isolated at previous levels…"

"The trans-operational stages are characterized by the evolution of structures whose internal relationships correspond to inter - operational transformations." (pp.173-184)

Despite its somewhat opaque language, the described process is not very complicated and its components can be traced out in the simple Paradigmatic Example below. The development of a concept, according to Piaget & Garcia (1989) starts with its isolated manifestations of the intra-stage. The comparison of isolated cases, search for similarities and differences based on the quality of discernment leads to the formulation of relationships between them. Formulations of relationships carries the inquirer into the inter – stage. Bisociations can be observed (1) during the formulation of these relationships and (2) during the transition from inter- to trans- stage as the moment of understanding joining all relationships developed in the previous stage into one structure. This leads us to conjecture that bisociation is a type of an instantaneous reflective abstraction – a mechanism of thinking formulated by Piaget involving the coordination of concepts, processes and entire structures to synthesize and develop new matrices of thought or schema. (Dubinsky,1991)

**Paradigmatic Example**

Consider the square root domain question in the classroom of a teacher researcher, demonstrating the interaction between student and instructor, in which the latter is able to get the student engaged in the thinking process and hence to facilitate student creativity. The domain of the function $\sqrt{X+3}$ is at the centre of the dialog.

Note that it is the spontaneous responses of the student from which the teacher-researcher creates/determines the next set of questions, thus balancing two frames of reference, his/her own mathematical knowledge and the direction taken by the student. Similarly the student has her own train of thought and prompted by the teacher-researcher's questions, she must now balance two frames of reference to determine her next response.

Note that Koestler would define a matrix as, "any pattern of behaviour governed by a code of fixed rules," (p.38) and the first line 1 the limitations of the students' internal matrix or problem representation is demonstrated. The teacher adjusting to the students' limited matrix provides two examples (line 6, 8) that provide a "perturbation" or catalyst for cognitive conflict and change, "...perturbation is one of the conditions that set the stage for cognitive change." (Von Glasersfeld, p.127).

In lines 6-9 the student reflects upon the results of the solution activity, through comparison of the results (records) they abstract a pattern, “the learners’ mental comparisons of the records allow for recognition of patterns” (Simon et al, 2004). Thus, in this example the synthesis of the student’s matrix for substitution and evaluation of algebraic expressions with their limited matrix of what constitutes an appropriate domain for radical functions (bisociation) resulted in the cognitive growth demonstrated in line 10.

In line 11,12 the perturbation brought about by the teacher’s question lead to the student entering into the second stage of the Piaget & Garcia Triad as they understood that the matrix or domain concept modifications for radical functions learned previously extended to this example as well. They were then able to reflect upon this pattern and abstract a general structural relationship in line 14 characteristic of the third stage of the Triad.

The problem starts with the function *f(X)* = $\sqrt{X+3}$

0 The teacher asked the students during the review: “Can all real values of be used for the domain of the function $\sqrt{X+3}$ ?”

1 Student: “No, negative $X$’s cannot be used.” (The student habitually confuses the general rule which states that for the function $\sqrt{X}$ only positive-valued can be used as the domain of definition, with the particular application of this rule to $\sqrt{X+3}$ .)

2 Teacher: “How about $X=-5$ ?”

3 Student: “No good.”

4 Teacher: “How about $X=-4$ ?”

5 Student: “No good either.”

6 Teacher: “How about $X=-3$ ?”

7 Student, after a minute of thought: “It works here.”

8 Teacher: “How about $X=-2$ ?”

9 Student: “It works here too.”

A moment later

10 Student adds: “Those $X$’s which are smaller than $-3$ can’t be used here.” (***Elimination of the habit through original creative generalization.***)

11 Teacher: "How about *g(X)* = $\sqrt{X-1}$ ?"

12 Student, after a minute of thought: "Smaller than 1 can't be used."

13 Teacher: "In that case, how about h(x) = $\sqrt{X-a}$ ?"

14 Student: "Smaller than "a" can't be used." (***Second creative generalization***)

The creativity of the teacher manifests itself in the scaffolding which led the student to the cognitive conflict between the two frames of reference. In the first case, the data driven results obtained through the matrix-process of substitution was synthesized with their limited matrix of the possible domain of a radical function, this bisociation and the resulting abstraction lead to a more complete understanding of the possible domain for specific functions. This represents a transition for the first to second stage of the Piaget and Garcia Triad. A continuation of this questioning process leads to further creative moments of understanding, in which the student was able to synthesize their understanding of the domain for two separate special cases of radical functions, this bisociation and the resulting abstraction into the structural understanding (line 14), suggests the student had crossed the ZPD from the second to third stage of the Triad.

We propose the method of scaffolding presented above as the teaching-research inspired guided discovery method of creating a bridge between Koestler's insistence on the "un-tutored' nature of bisociation with Vygotsky emphasis on the socially structured nature of learning environment.

**Concept Formation within Problem Solving: Vygotsky & Koestler**

Vygotsky, like Koestler treats learning, in particular concept development within the framework of problem solving, "concept formation…is an aim directed process…for the process to begin, a problem must arise that cannot be solved otherwise than through the formation of a new concept (Vygotsky, 1997, p.100). The problem solving environment that leads to concept formation is described by Vygotsky using the work of the Gestalt psychologist Max Wertheimer. In this framework concept formation results from productive thought which, "is based upon insight i.e. instant transformation of the field of thought. The problem X that is a subject of our thought must be transformed from the structure A within which it has been first apprehended to the entirely different structure B, one must transcend the given structural bonds and this […] requires shifting to a plane of greater generality, to a concept subsuming and governing both A and B" (Vygotsky,1997, p.205).

The distinction that Koestler, and Vygotksy make between routine and insight problems solving is the degree of consciousness or awareness required. For Vygotsky and followers of Piaget cognitive change involves reflection and the abstraction of patterns for actions and their effects. Thus, cognitive change (accommodation) requires first cognitive conflict as an individual applies their existing matrices to a new problem situation, i.e. it is triggered whenever "an individual try to understand, organize or make sense of a new situation" (Steele & Johanning, 2004) and second, the situation requires synthesis of a new concept with this matrix. As this synthesis requires judgments of comparison and differentiation on how the new concept relates to this matrix, this process breaks the routine automatic thinking of proceduralization and can lead to generalization of the matrix and concept formation. "But in original discoveries, no single pre-fabricated matrix is adequate to bridge the gap. There may be some similarities with past situations, but these may be more misleading than helpful […] here the only salvation lies in hitting upon an auxiliary matrix in a previously unrelated field one may call this […] reasoning from a parallel case." (Koestler, p.201)

Vygotsky would allow for concept formation or cognitive change with the help of instructor's (or a peer group's) facilitation of understanding while Koestler would make a subjective distinction between such tutored instruction and untutored learning. Koestler would allow the synthesis of concepts related to previously unrelated matrix-structures to be considered as 'creative' only when this synthesis occurred organically, without structural guidance i.e. untutored learning. "Familiar situations can be dealt with by habitual methods; they can be recognized at a glance as analogous in some essential aspect to past experiences which provides a ready-made rule to apply to apply to it." In contrast, problems that are unfamiliar to the solver provide the opportunity creativity

> "the more new features as task contains the more difficult it will be to find the relevant analogy, and thereby the appropriate code to apply to it…one of the basic mechanisms of the Eureka process is the discovery of a hidden analogy; but hiddennes is again a matter of degrees. How hidden is a hidden analogy" (Koestler, p. 653).

An example that involves a synthesis of matrices in elementary algebra is the use of fractional exponents to represent roots. In this situation, the matrix or structure associated with fraction notation is used in an entirely new situation that has little to do with the concepts of part-whole, ratio, quotient or even the fraction as numerical value on the number line (measure) concepts that typically underlie fraction notation. Thus, both Koestler and Vygotsky would view a tutored social situation presenting the topic of fractional exponents for the first time as leading to concept formation however, Koestler as a constructivist would insist that originality and creativity would only come when the situation was untutored i.e. learning based upon student discovery.

In contrast to Koestler, Vygotsky has a strong focus on the role of education in concept-schema development, "school instruction induces the generalizing kind of perception and thus plays a decisive role in making the child conscious of his own mental processes. Scientific concepts, with their hierarchical system of interrelationships, seem to be the medium within which awareness and mastery first develop" (Vygotsky, p.171). This consciousness as noted by Vygotsky arises with adolescence, during the middle school years of education when students are required to and many struggles to learn fractions, proportional and algebraic reasoning.

The role of education in Vygotsky's framework is to present problems on the upper structural level of the individual's ZPD and then provide them with guidance in reaching that goal. To the constructivist like Koestler who believes that learning only has meaning when the individual reinvents or relives the process of discovery, Vygotsky would counter that guided learning even when students follow instruction without grasping the essential processes is valuable within the individual's ZPD. "To imitate, it is necessary to possess the means of stepping from something one knows to something new."

Our Teaching Research approach to the Koestler/Vygotsky contradiction is to find a compromise between the two positions. As the previous section Paradigmatic Example suggests, the Teaching Research methodology incorporates successful facilitation of bisociation into student-teacher interaction during which student's ZPD of the concept is traversed from the initial to the final stages of the Piaget & Garcia Triad. Thus, it is possible to facilitate student creativity and grasping conceptual connections in the context of guided discovery method within student ZPD.

In this situation, the synthesis of student's matrices seen as their internal representation of the domain of a radical function with their reflection and abstraction required to resolve the cognitive conflict brought about by the teacher's data driven examples.

The reflection required to accommodate this new information for the specific radical function analysed and the resulting conceptual awareness of the effect the numerical values under the radical had on the domain of the function were then transferred to another similar function and their understanding of these situations was synthesized to form an abstract new structural understanding in the manner Vygotsky describes concept development.

**Conclusion**

The aim of this report has been to show the mutual interactions between the theory of Act of Creation and learning theories in mathematics education of Piaget, and Vygotsky, in explaining concept and schema development within a problem solving environment. The integration of their work focuses on bisociation as a mechanism of conscious reflection to promote concept development in Vygotsky's theory and as a type of reflection and abstraction i.e. the mechanism of growth through the Piaget & Garcia Triad for schema development. As such this article provides impetus for further research to integrate creativity into mathematics education and comparative analysis of the mechanisms of learning based upon reflection and abstraction with bisociation.

One of the central issues to be resolved in the process of integrating Koestler and Vygotsky approaches is in the interaction of social environment of instruction with bisociation discussed above. We have provided an example a teaching-research method for such scaffolding the teacher-student interaction, which leaves the student with sufficient amount of intellectual freedom to facilitate a creative leap of insight, the bisociation. This example suggests a more general question: how to construct such instructional dialogues, which provides for an optimal amount of student intellectual freedom to promote the occurrence of this creative leap of insight?

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# Brief creative assignments in undergraduate mathematics courses: Calculus 3 and Linear Algebra

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***Abstract**: This presentation contains brief creative assignments about mathematical convention and notation, real life examples, mathematical examples, mathematical terminology, and inquiries about mathematicians and their theorems. The theory of creativity that supports the concept of creative assignments in the classroom is supported by Graham Wallas (1926) description of stages of thought and by other well established psychological theories. Those assignments are serving as pilots for longer creative assignments and research projects aimed for improving students' satisfaction and performance in Calculus 3 and Linear Algebra. In addition, the creative assignments may improve students' graduation and retention of the underserved and underrepresented students in STEM. The classes where the assignments took place were taught at a large urban community college in small classrooms of ten to twenty-five students. The assessment is based on students' reactions in the classroom and on students' responses to a survey conducted at the end of the semester.*

*Keywords*: Calculus 3, Linear Algebra, creativity, creative assignments

## Introduction

Creativity is probably the last concept that comes to mind when thinking about teaching or learning undergraduate courses such as Calculus 3 or Linear Algebra. This is a sad statement that undergraduate curriculum does not nurture knowledge by emphasizing creative assignments, but simply provides the material to be mastered. Unfortunately, without appropriate spirit of inquiry and without applicable context, the knowledge learned in classroom disappears within short time: often not building enough foundations even for handling comprehensive final exam questions and not building knowledge throughout multiple courses and semesters.

But it does not have to be that way. If instead of following the outline of the lecture rigorously, we introduce brief creative assignments in the classroom to bring educational enjoyment equally to the teacher and to the students. The goal of this enjoyment is to improve class atmosphere and student satisfaction but at the same time keep the quality of students' work, high standards of course outcomes, and integrity of students' knowledge. This article contains a few examples of such assignments to serve as inspiration for future course improvements.

## Traditional Theory of Creativity

What is creativity? We are the most creative when we are children yet there is usually no scientific research involved. What is important is that being creative and playful is probably the most satisfying among all attitudes which somehow gets lost when we grow up.

The formal definition of creativity, or rather the lack of it, was presented in detail by Runco and Jaeger[1]. From the perspective of mathematical classroom, creativity happens when new things are created, or more precisely, creativity is a mental process that transforms old ideas, rules, patterns, relationships, etc., into relevant new ideas, forms, methods, interpretations, etc. For the purpose of this presentation, a brief creative assignment in is defined as a 15-minute question directed to students that reflects on the current class material and contains an element of students reaching towards subconscious and reflections on the outcomes.

The deep reasons for the need of creative assignments in the classroom was discussed by Savic[6]. Creative assignments may touch various aspects of the course. The main purpose of assignments presented here is to expose students to the joy of tapping into the subconscious while consciously following the lecture.

The benefits of such assignments include bringing joy, interest and students' involvement to the course. This helps observing students' attitudes towards creativity and unfettered thinking. At the same time, it provides assessment of actual connections among mathematical topics that students have built in their minds.

## Stages of creativity

In his book "The Art of Thought" Graham Wallas[7] underlines the necessity of observing the thinking process while learning the actual subject (page 28):

> But behind the use of thinkers of rules and materials drawn from the sciences there has always been, since the dawn of civilization, an unformulated "mystery" of thought which has been "explained" by no science, and has been independently discovered, lost, and rediscovered, by successive creative thinkers. [Who] learnt from each other something which was neither logic nor accumulated knowledge.

This "mystery" from the quotation above describes connections between subconscious and conscious mental formations generated during the process of learning, and outside of it. Wallas indicates four stages of though: Preparation stage, Incubation stage, Illumination stage, Verification stage. The Preparation stage occurs when a student is learning about the problem. The Incubation stage happens when the subconscious mind is working on the inquiry. The most mysterious is the Illumination stage where the subconscious mind reveals its findings to the conscious mind and brings it to the attention of the student. The Verification stage occurs when the conscious mind analyzes the provided results with the conscious mind and verifies the results.

One observes that the process of problem-solving consists of multiple repetitions of those four stages. Indeed, anybody who ever solved a basic problem knows that the first solution provided by the subconscious mind may be incorrect or incomplete and the entire process becomes more of a cycle, where collecting information and incubating the answers, then illuminating them, and eventually revealing to the conscious mind, follow one another multiple times. An example of such a sequence encouraged by a mentor is, for example, revealed in "Minutes from a math meeting with an undergraduate student."[3] The focused state that involves those stages was already recognized and described as Creative Flow by Csikszentmihalyi.[1]

**When the cycle does not work?**

The theory brought by Wallas[7] and discussed later by multiple authors of psychology and philosophy, is not entirely aligned with the modern theories of the Inquiry Based Learning. But indeed, it simply assumes that the first stage "Preparation" is experienced effortlessly and naturally. However, without initial intentions for learning, or questions in mind, or desires for discoveries, this first stage would not take place. That is why the classrooms may be filled with students having no curiosity in their minds and who are not ready for the preparation stage. It is clear that without preparation stage the other stages will not take place.

If the cycle functions properly, it brings a sequence of satisfying events of inquiry, preparation, incubation, illumination, and verification followed by more inquiry, etc. However, if the cycle does not function, where shall we, as teachers and mentors, address the difficulty? Before responding to this question, one needs to observe that a deep inquiry is a subconscious stage, just like the stages of incubation and illumination. Thus, the process of creation is an internal discussion between conscious and subconscious states. When the cycle does not work, one would like to find methods of addressing the difficulty.

Wallas mentions mental tension between conscious and subconscious (page 34) and points out that this phenomenon was already discussed by H. Poincaré in *Science and Method*[4], who calls the force *'sensibilite'* translated as *'feeling.'* Is then this feeling that needs to be addressed in the classroom when students experience difficulties with solving problems or paying attention? Wallas, however, gives little hope for investigating methods of successful improvement of the process of creation in terms of doing more or working harder. He even indicates that being more conscientious of the process of thinking may disturb the thoughts and produce no results whatsoever.

## Sample Assignments

This section contains sample assignments.

**Justifying mathematical convention in Calculus 3 class**

Often Calculus 3 students have difficulties understanding why the 3-dimensional space is represented with the $yz$-plane being drawn as the plane of the board or paper and the $x$-axis coming out of the board with the positive direction pointing towards the observer. To address this issue, I introduced a brief creative assignment, where each student made few sketches with suggestions how the three-dimensional space could be drawn. Then students presented their work on the board and plotted the point $(1,2,3)$. When the pictures were compared, it become evident that prior to understand each other's drawings at the first glance, we have to stay within one convention. At the same time students recognized positively and negatively oriented 3-dimensional spaces. Group work is very useful for this particular activity since students can look at each other's drawings to compare and verify whether two pictures represent exactly the same 3-dimensional space or its mirror image. The activity took about 15 minutes of class time. Figure 1 contains sample sketches.

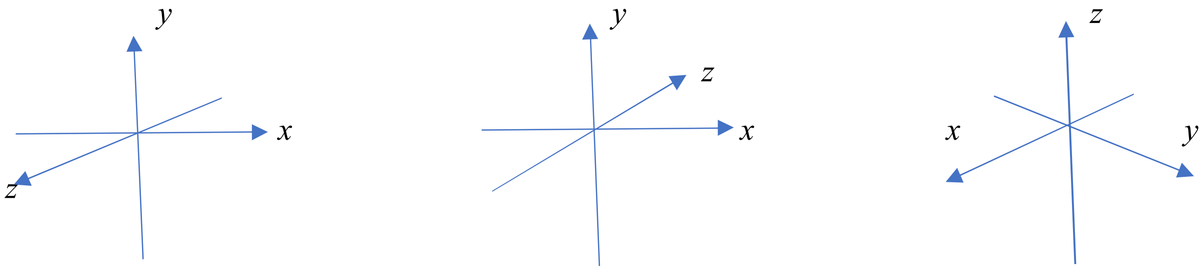


Figure 1. Sample students' work: positively and negatively oriented 3-dimensional spaces.

**Justifying mathematical terminology in Linear Algebra**

Usually terminology of math concepts reflects on their connections with life examples, name of a person honored or simply another idea. The general motivation for providing appropriate math terminology is to create a connection of the concept and vocabulary. But sometimes those ideas may be so particular and exquisite that modern education does not furnish students with suitable knowledge necessary to make this connection. In such a case, math terminology may become a true burden unless it is used as an excuse to expose students to new topics. The **echelon** form of a matrix is an example of perfectly placed similarity between the appearance of a reduced matrix and military formations known already in the Roman times. A brief creative assignment involves students searching on their cellphones for answers of few questions about the right echelon form, the left echelon form and their relationship to matrices. Students are asked to prepare a brief description what "echelon" means and should visualize their presentation with a picture or a photo as presented on Figure 2. The activity can be performed in class or assigned as homework.

Figure 2: Presentation of the echelon formation.[2]

**Providing real life examples in Calculus 3 class**

Frequently teachers of mathematics receive requests from students to clearly state the life applications of presented methods, concepts and theorems. Classroom methodology of responding to such an inquiry, involves asking students what examples they can provide based on their own experience. When introducing quadric surfaces and parametric curves, my students spend time in the classroom finding familiar shapes that are similar to those presented on the screen. Usually the examples are divided into those that appear in nature and those that are man-made. The most frequently analyzed curves are the helix and the spiral. When looking at them students recall car shocks, springs inside pens or side-spiral college notebooks. Often students while analyzing the physical models of the spirals claim that there is only one spiral, it just needs to "turned around." But after rotating the model several times, they eventually arrive to the illuminating conclusion that the two spirals are not identical. Figure 3 presents examples of different spirals.

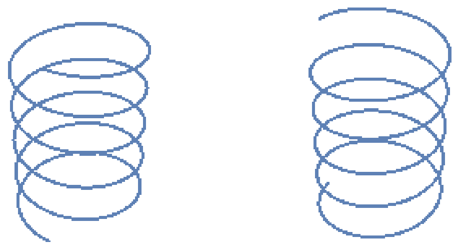

Figure 3: Samples of right and left sided spirals.

Among the surfaces the saddle is the most difficult to sketch but provides the most satisfying examples in a form of Pringles potato chips. When sketching a shape of a paraboloid $z = 8 - x^2 - y^2$ trimmed to the region $-1 \leq x \leq 1, 0 \leq y \leq 2$, a student realized that the graph reminds him of a segment of a parachute.

Similarly, when introducing triple integrals in cylindrical and spherical coordinates, we sketched a picture of the solid that lies outside of the cylinder $x^2 + y^2 = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$. Students were asked what real-life shapes come to their mind when looking at this solid. Students pointed out that some beads are shaped just like the solid. Figure 4 presents the surfaces and the solid considered in class.

Figure 4: Pringles surface, parachute-like surface, and bead-like solid.

**Providing mathematical examples of typical regions in Calculus 3 class**

In the past, before I made a commitment of introducing creative assignments in my classes, I prepared a neatly organized sequence of examples to introduce students to a variety of typical regions. Then I presented those examples in class and asked students to describe boundaries of the regions. When my perspective changed, the definition was followed by one example and then I asked students to produce their own examples. Students worked on the assignment for about 15 minutes and after receiving my feedback and encouragement, presented their work on the board. While observing the class I realized that some students immediately began producing examples but some had no trust in their mind and simply did not know what region to think of. I directed those students to the recent worksheet with an example of a domain of a function. As I observed, most students who began working promptly, started with a region in their mind and then tried to obtain functions that match the shape of their example. After reviewing lots of students' examples I realized that the variety, as presented in Figure 5 was just as good as the lecture prepared by me. At the end of the activity students sketched their regions on the board and provided their solutions. Then I analyzed all solutions and corrected the mistakes on the board. Students seemed to be more interested and involved in the topic than usual.

Figure 5: Sample typical regions provided by students: rhombus, trapezoid, semicircle, region between two circles.

**Generating inquiries about mathematicians and their theorems mentioned in the textbook of Calculus 3 class: Fubini, Stokes, Green.**

Doubting that a plain biography of a mathematician is placed properly in calculus class for engineers, I introduced creative assignments for students to become familiar with the historical background and personal life of few mathematicians mentioned in the textbooks. Sometimes the discovery when and where a particular mathematician lived brings additional questions. Students often realize that some theorems and observations are relatively young just like Fubini's Theorem from 1907, some other theorems are "slightly" older just like Green's Theorem from 1828, comparing to the concept of the tangent line already described by Euclid in 300 BC. The most surprising discussions are often created by the simplest and most innocent questions like "Where is Glasgow?" Since students represent highly heterogeneous group with various cultural and geographical experience, they often amaze each other with their questions and findings.

**Observations and Challenges**

While observing students during their creative work I realized that some of them did not get involved with the question and did experience the illumination stage. However, I would still consider the assignment successful from the perspective of those students' point of view since they were somehow exposed to the process when observing their peers going through all four stages of creativity. Moreover, the process was repeated multiple times in the classroom, and thus the observations were recurrent. I hope that during those brief assignments, the classroom offered safe and friendly environment for students to encourage and practice the habit of creativity while learning the material. Some students experienced illumination stage which was visible even from a distance in a form of a facial expression or even as a subtle jump of energy within the body.

As a teacher and a mentor of multiple research projects, I have been searching for methods of gently tapping into of students' subconscious minds to reveal the hidden potential of creativity. Personally, I believe that the habit of creativity can be trained just like other habits of mind. But the training requires suitable environment and appropriate circumstances to produce a genuinely creative illumination.

## Assessment

Assessments for brief creative assignments are based on students' formal and informal responses. During the informal assessment proctored immediately after the brief creative assignment in Calculus 3 classes, I asked students whether they would prefer less or more of those assignments. Students agreed that they found this activity valuable and encouraging. During the formal assessment, I asked students from my Calculus 3 classes for feedback about all class modules: mini-lectures, worksheets, solving problems on the board, cardboard projects, etc. One of the questions asked for students' opinion about brief creative assignments. All students gave excellent marks to this assignment and did not suggest any improvements. However, since the sample group was very small and consisted of only five students, this assessment needs to be repeated for a larger class.

## Conclusions

Students' involvement, their reactions to the brief creative assignments and the results of the survey indicate that this type of class activities are welcome in the classroom with joy. However, the most important benefit of the creative assignments comes from binding the conscious and subconscious mind by creating an everlasting conversation among their mental formations. Those conversations may seem insignificant to the entire course curriculum and marginal from the perspective of learning calculus but they are important or even crucial for the unity of knowledge.

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# "Least Squares Estimation" instructional design based upon APOS theory: Laying Mathematical Representation and Transformation Bridge

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***Abstract:*** *The teaching design of "least squares estimation" under the framework of APOS theory is discussed in this paper. The main points of instructional design is to create the sequence of that several mathematical representations and transformations in the four teaching phases of operation, process, object and schema. Different mathematical representations can not only connect four stages of teaching, but also achieve the mathematical transformation of the final statistical problems from the "uncertainty" to a linear function of " certainty" change, so that students really understand the idea of least squares, and the formation of Schemata of the idea of linear regression, between " algebraic expressions" and " geometric meanings"; between different " algebraic expressions".*

**Keywords:** APOS theory; least squares estimation; teaching design; mathematical representation and transformation.

## The question is raised

The content standard set by the Ministry of Education in the "General High School Mathematics Curriculum Standard (Experiment)" (hereinafter referred to as "Mathematical Curriculum Standards") published by the Ministry of Education in April 2003 is: students learn

a) to estimate the overall population and its characteristics of thought through learning random sampling, sample estimation overall, linear regression and other basic methods;
b) through systematic experience of the entire process of data collection and processing, they learn statistical thinking and deterministic thinking differences.

So, what is statistical thinking? Dan Zhang believes that statistical educators and statisticians depict the students' statistical thinking in two aspects: (1) the statistical process and the statistical method level, which is reflected in the following aspects: "to collect information through data collection and analysis and to understand the randomness through data, and (2) to use statistics to explain phenomena and solve practical problems. This level is reflected in "using data to solve problems (including data awareness)" (DanZhang, 2010). Ningzhong Shi believes that the statistical process should include two core features: extracting information through data analysis; and understanding randomness through data. The randomness of data means that for the same thing, the data collected each time may be different; as long as there is enough data, we can find out the law (Ningzhong Shi et al., 2005). Dan Zhang summary randomness of data means: First, uncertainty, and second, the regularity of sufficient data (Dan Zhang, 2010).

From the above perspectives of DanZhang and NingzhongShi, one of the important characteristics of the statistical problem is the randomness of the data. This "randomness" is the manifestation of "uncertainty." This "uncertainty" is reflected in the relationship between the two sets of data for bivariate data in life such as height and length; age and blood lipids, but this relationship cannot be determined by a definite functional relationship. The essence of a functional relationship is to determine the relationship between independent variables and dependent variables is determined, such as if $x_1=1$, $y_1=f(x_1)=1$, $x_2=2$, $y_2=f(x_2)=4$, $x_3=3$, $y_3=f(x_3)=9$, then the correspondence between the domain $(1 \le i \le 3, i \in Z_+)$ and the values in the range can be functionally determined as a quadratic relationship that provides an element of certainty to the original randomness. The standard of mathematics curriculum believes that the teaching of high school statistics aims to improve students' ability of data analysis. The manifestation of data analysis ability is whether one grasps the ideas of random sampling, sample estimation population, linear regression, and independence test and regression analysis method.

Students can not only understand the uncertainty of statistical data while learning linear regression, but also know that the law of implication can be found out from the uncertainty of data. The essence of this law is the "certainty" of the functional relationship. Therefore, in the teaching of linear regression, the following questions need to be solved: What is the "uncertainty" characteristic of the data? How to find out the law that it contains? After finding out this law how to explain this law and use this law?

**The content of the study**

Textbook referenced in this text is "Mathematics 3 (Required)" (hereinafter referred to as "compulsory 3"), an experimental textbook of ordinary high school curriculum standard published by Beijing Normal University Press, which is matched with "Mathematics Curriculum Standard". The topic of "Linear regression" is not designed as a single section in the first chapter of "Compulsory 3", but consists of Section 7 "Relevance" and Section 8 "Least Squares Estimation". The content requirements of the "Mathematics Curriculum Standards" on the "variable relevance" is:

(1) "by collecting real data of two variables associated with the scatter plot, and the use of scatter plot intuitive understanding of the correlation between variables' relationship ". The students, in the process of collecting data and drawing a scatter diagram can realize that the relationship between these two variables is not a "certain relationship", because it cannot be described by the functional relationship; one can only say that the two variables are related.

(2) "to experience the use of different estimation methods describing the two variables linearly related; to experience the process of knowing the idea of the least squares method, when linear regression equation can be established according to the linear regression coefficient equation." The linear regression equation is the "law" found from the correlation between these two variables. However, the essence of this equation is a linear function. Since it is a functional relation, the relation between variables is then "definite". So why is the relationship between the two variables that are not "definite" finally "determined"? The idea of the least-squares method embodies the process of "determining" the relationship.

The purpose of this paper is to explore the teaching of least-squares estimation using APOS theory and reveal how least-squares transform "uncertainty" into "certainty."

**Literature review**

The same mathematical concept often shows the characteristics of duality: it is both a process operation and an object structure; it is both an operation behavior and a structural relationship; both dynamic and static characteristics (Weiping Zhang, 2014). As the concept of process is related to APOS theory (Tall, Gray, 2001; Tall, 2006), APOS theory can be used to guide the learning and teaching of statistical knowledge.

The theory of APOS was first put forward by Ed Dubinsky of Georgia State University on the theory of Reflecting Abstraction. Shiqi Li thinks that to reflect on the abstract is to do some practical activities, then take a step back and review one's own practice activities to consider one's activities in the position to be considered. Then the activity becomes the object of thinking and the related abstraction is completed. Activity experience only provides the basis for the concept of organization, and the activity itself does not provide a concept (Shiqi Li, 1996).

APOS is the acronym for the four English words Action, Process, Object, and Schemas respectively. According to the theory, learners can obtain and establish the connection between knowledge based on reflection and construction after learning the mathematics through several stages of operation, process and object (LijuanCao , 2012). ShiqiLi think mathematics learning is an empirical activity. He believes that the recognition of mathematical objects must first have the basic behavior of mathematical cognition, namely "operational computing", with the "operational computing" experience after the realization of mathematical objects may be further manipulated (Shiqi Li, 1996).

Therefore, "action stage" refers to the "operation" the student experiences. This "operation " is not only "operation" in the narrow sense but also all mathematical activities in a broad sense, including conjecture, memory, calculation, reasoning, observation graphics (Weiping Zhang, 2014). This "operational computing" phase is the "activity phase," where students experience these maths and experience the relationship between the intuitive background and concepts of mathematics (Xuefen Gao, 2013). The "process stage" is the process of thinking about "activity", experiencing the internalized process of compressing the mind, describing the activity in the mind, and abstracting the concept-specific nature (Xuefen Gao, 2013).

When "activity" repeats and leads to constant reflection on the part of learners it establishes an inner mental structure that can perform similar activities without external stimuli (Weiping Zhang, 2014). At this stage, learners form programs that do not need to be manipulated in their minds and can reverse the program or combine it with other programs to form procedures, steps and abstraction of the peculiar nature of the concept (Lijuan Cao, 2012). After the learner has experienced the essence of the concept through the "abstraction" of the previous stage, the formalized definition and symbol of the concept are given as a concrete object. This stage is the "stage of the object" (Xuefen Gao, 2013). After "process" is treated as a whole and can be transformed, "process" condenses into "objects" (Weiping Zhang, 2014).

Learners establish a connection with other concepts, rules, graphics, etc. The stage in which the mental schemas form in the mind is the "schemata" stage (Xuefen Gao, 2013). At this stage learners build a coordinated knowledge network that remembers the problem context associated with the concept (Weiping Zhang, 2014).

As can be seen from the above, the four phases of APOS can interpenetrate each other, and the boundaries of distinction are not necessarily clear, but gradually transiting. For example, repetition of activities can lead learners to reflect on "activities" and then to transit to the "process stage", that is, some "activities" belong to the "operational phase" while others are “Process stage ". The criteria for this division are: whether the sequence of activities can lead to reflection on the activity or not; entering the "object stage" after formalizing the definition and the sign of the essence of the concept recognized in the "process stage". After formal definition and symbol of the concept is recognized in the process stage, it enters the" object stage ". During the development of understanding complex concepts," process stage "and" object stage " will be often repeated.

Statistical knowledge has process characteristics, which can be coupled with the APOS theory and procedural concepts so teaching statistics can utilize APOS theory.

**The research process**

The following are four stages of APOS theory to analyze and design "least squares estimation" teaching:

(1) Activity stage. At this stage, one of the activities of teachers and students is to review the knowledge of the last lesson. As students learned in Section 7, "Relevance," students already know that some of the bivariate data in their life have a linear correlation, reflected in the image that the scatter points are not in a straight line, but appear to be near a straight line. At this stage, one of the activities of teachers and students is reviewing the knowledge of the last lesson. The examples of activity stage are questions to be solved by "least square estimation": "what kind of line can best

characterize the scatter plot", and "How to find this line?" Therefore, at this stage, the second part of the activities of teachers and students is to observe the graphics, and guess the conclusion. These two teacher-student activities provide the foundation for the next phase of the organization's concept. Observing the graph, the conjecture conclusion itself does not provide yet a concept, and therefore these two activities are included in the activity phase.

(2) Process stage. Here we seek the connections between different aspects of the concept. According to the previous literature review, we can see that the teaching activities in the activity stage and the process stage may be similar. Then one criterion for judging whether the teaching activities are in the process is whether the activity provoked students' reflection. For the first question, students may have different answers, teachers can guide students to think about the discussion and find out that if the scatter points are all near the straight line. If not all the points fall on the straight line, the more suitable straight line should satisfy the diagram. "The distance from the point to the straight line is all 'close' enough so that there is a need to express the "point-to-line distance". Since the straight line is indefinite, the coefficients in the line equation $y = bx + a$ are to be determined. In fact, the process of determining the coefficient and value is the solution to the second problem.

When the distance of the scatter A($x_i$, $y_i$) to the straight line is expressed as follows: $d = \frac{|bx_i - y_i + a|}{\sqrt{b^2+1}}$, we can see that the minimum is the minimum. Then the next question is: If the suspected minimum d= $\frac{|bx_i - y_i + a|}{\sqrt{b^2+1}}$ is $|bx_i - y_i + a|$ really the smallest? If so, then thinking back to the question also leads to: if the suspected $\frac{|bx_i - y_i + a|}{\sqrt{b^2+1}}$ is minimum whether the distance $|bx_i - y_i + a|$ is the minimum, and therefore the smallest distance? The conclusion drawn here is a very crucial step.

(3) Object stage. According to the previous literature review, when the learner has experienced the essence of the concept through the "abstraction" of the previous stage, the formalized definition of the concept and the symbol given to it become a concrete object. This

stage is the "object stage" (Xuefen Gao, 2013). The key question to understand the Least Squares method of thinking is why you use "deviation" to mean "point-to-line distance." Therefore, "deviation" is the "object" of understanding, and students' understanding of this concept cannot appear out of nowhere. The next (and very easy to overlook) question is: in case $\frac{|bx_i - y_i + a|}{\sqrt{b^2+1}}$ is min, whether $|bx_i - y_i + a|$ is also minimum? If so, then the next (and very easy to overlook) problem is: What is the geometric meaning of $|bx_i - y_i + a|$?

In other words, if $|bx_i - y_i + a| = |y_i - (bx_i + a)|$ from the images (see below (1)) can be seen, $|y_i - (bx_i + a)|$ that is the distance of AB. So, the sum of the distances of the scatter points to the straight line is the minimum. The sum of the vertical distance AB (vertical segment AB is also called deviation) is the smallest possible. In terms of the geometric meaning to this formula, then the expression expressed by this formula is the shortest distance from point to line.

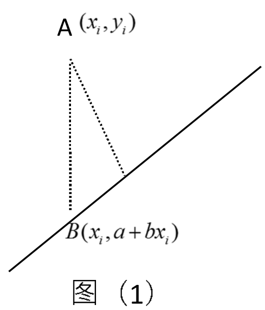



**FIG. 1**

The next problem is that the sum of the accumulated deviations involves the sum of the absolute values, and the square of the sum of the deviations can be summed up in order to remove the sign of the absolute

value that is advantageous for the additive deviations without any positive or negative cancellation. Then the final determination of the coefficient and value is to consider the square of the square of the deviation as a quadratic function and to use the knowledge of the quadratic function to solve it. Derivation formula, "mathematics curriculum standards" and "compulsory 3" are not required, but recommended for interested students to try derivation.

(4) Schematic stage. Through the first three stages of learning, students can form a psychological schemata around the core issues are: point-to-line distance can be described as "deviations", the shortest deviation is the point-to-line distance is the shortest. According to the previous analysis, this straight line can satisfy the fact that the distance between the existing scatter point and this straight line is "minimum", and the essence of "minimum distance" is " Minimum error ", so if the linear error based on the existing data to obtain the smallest, then according to this line to make the prediction error will be the smallest. In addition, with least-squares estimation, students develop a larger mental schemata that knows how to use a straight line to fit a scatterplot and find an "optimal" line that makes the line and scatter.

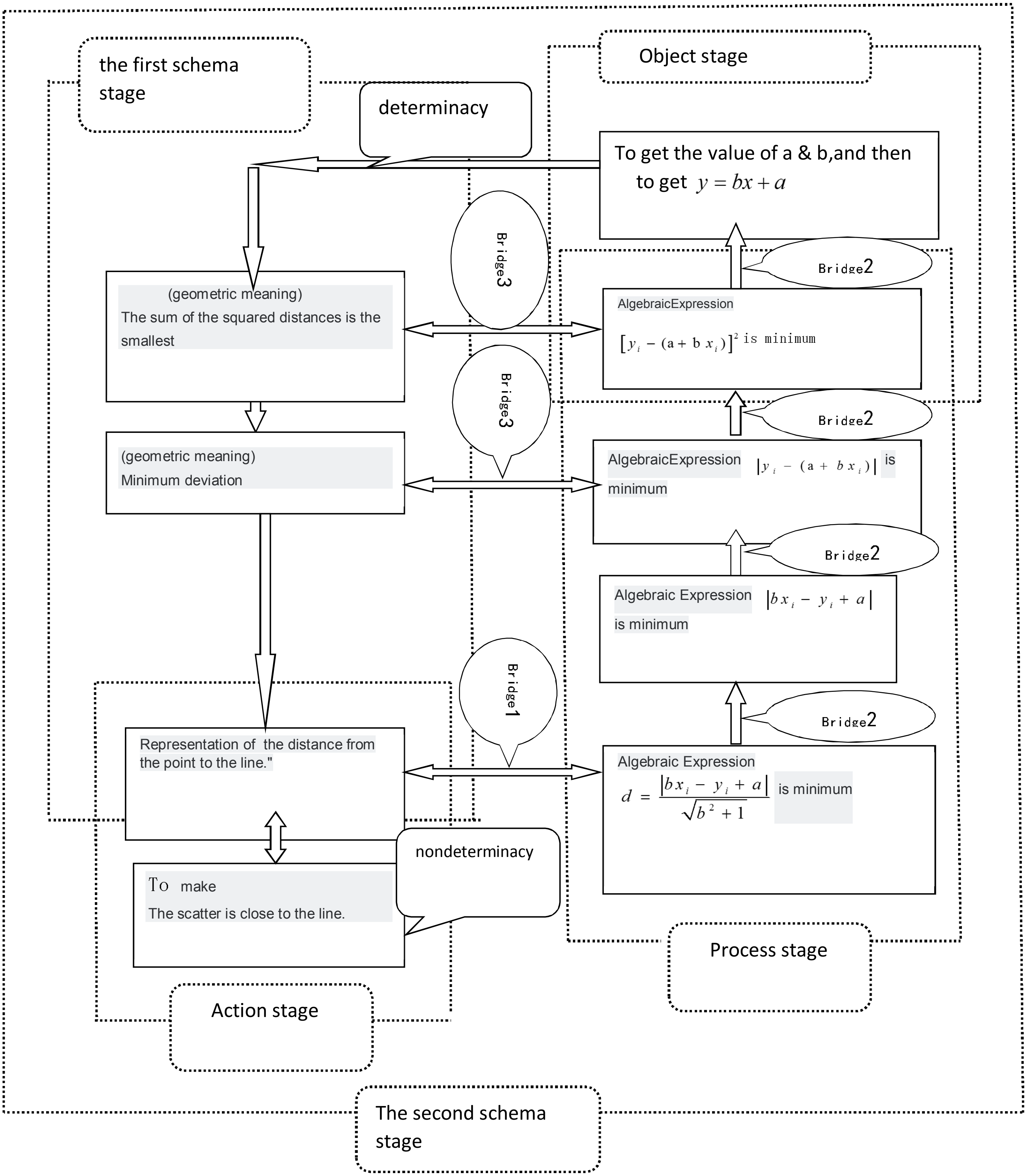
**The conclusion of the research**

From the above analysis point of view, to enable students to establish the link from "uncertainty" to "certainty", the key is to set up several bridges of mathematical representation and transformation, so that students gradually establish experience through activities, processes; After a period, the final form of linear regression method becomes clear. These bridges are (see Figure (2)).

Bridge One is between the "problem or the target to be solved" and the "algebraic expression," and to approximate the scatter to a straight line, describe the mathematical concept of "proximity" as "distance", and is "point-to-line distance," so the process of mathematical representation from "problem or goal to be solved" to "point-to-line distance".

Bridge two is between different "algebraic expressions", the process has two mathematical transformations,

The first math transformation is from $\frac{|bx_i - y_i + a|}{\sqrt{b^2+1}}$ to $|bx_i - y_i + a|$, the difference between the two algebraic expressions is whether to consider the denominator. It is a complicated process;


the first schema stage
Object stage
determinacy
To get the value of a & b,and then to get $y = bx + a$
Bridge3
Bridge2
(geometric meaning)
The sum of the squared distances is the smallest
AlgebraicExpression
$[y_i - (a + b x_i)]^2$ is minimum
Bridge3
Bridge2
(geometric meaning)
Minimum deviation
AlgebraicExpression $|y_i - (a + b x_i)|$ is minimum
Bridge2
Algebraic Expression $|bx_i - y_i + a|$ is minimum
Bridge1
Bridge2
Representation of the distance from the point to the line."
Algebraic Expression
$d = \frac{|bx_i - y_i + a|}{\sqrt{b^2 + 1}}$ is minimum
nondeterminacy
To make
The scatter is close to the line.
Process stage
Action stage
The second schema stage

The second mathematical transformation is from $\left|bx_i - y_i + a\right|$ to $\left(bx_i - y_i + a\right)^2$.

The process, to find the absolute value of the problem cannot be a linear operation.

Bridge three is in the "algebraic expression" and "geometric meaning" between the mathematical characterization process $\left|bx_i - y_i + a\right|$ expressed as $\left|y_i - (bx_i + a)\right|$. After that, it can be understood as "dispersion", which is the process of interpreting the algebraic interpretation into a geometric meaning. In the learning process of least square estimation, students can form two schema structures. The first schema structure is between "point-to-line distance" and "dispersion" and "square of dispersion", but this structure must be in the operating stage, the process stage, so that the object stage can be established, or else students cannot understand why the expression of scattered points and the distance from the straight line to represent "square of dispersion".

The second schema connection is from the whole process of thinking between "uncertainty" to "certainness". After forming this schema structure, students can understand the nature of the linear regression method and can use this method to reason and predict the data. Because this part of the score is not high, not a college entrance examination hot spots, many teachers are generally doing simple teaching, teaching most teachers will not let students experience "with" deviation "to characterize the scattered point to the 'distance'. They think that as long as students know the formula for calculating the sum of the coefficients in the regression equation, they will imitate the training (YangchunXie, 2016). The result of this teaching is that students do not really understand the essence of the least-squares method.

Shiqi Li think mathematics object is a kind of thinking object, the mathematical content can be divided into two types of processes and concepts. The establishment of mathematical concepts and methods is, in essence, a process of reflection and construction on the basis of experience, on the basis of experience, and lacking in operational procedures, so that the process of this activity and operation must be personally experienced by the students, (Shiqi Li,1996).

"Least Squares Estimation" realizes the transformation between the "uncertainty" (or "randomness") of statistical problems and the "certainty" of functional relationships. The establishment of this method of thinking requires that after experiencing the four Therefore, when teaching, teachers need to build a bridge of understanding for students through mathematical representation and transformation and care for the integrity of the learning process. In particular, teachers should pay attention to the teaching of "process stage" Stage of mathematical representation and thinking about the purpose of mathematical transformation, the lack of this stage will lead to student thinking "chain" fracture, not only will affect the formation of mathematical characterization and transformation of students, but also further affect their knowledge and experience will eventually Affect the real formation of the schema structure.

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