

## Using Calculators in Teaching Calculus

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### Abstract

*This paper presents some ideas of using an advanced scientific calculator in teaching of Calculus. We will illustrate our approaches on example of CASIO fx-991 ES PLUS calculator, widely used by the students of our college. The advanced features of this calculator make it possible an effective use not only as a tool for quick computations but as a helpful instrument for enhancing understanding of main concepts and algorithms of calculus, for development of explorative, critical and creative thinking of the students. We discuss also how symbolic calculus ideas may be motivated and evaluated by means of calculator. What's more we describe how the calculator itself may be investigated and what are calculator's restrictions and mistakes. We indicate how people's thought may be better than calculator in some computation problems. As it is mentioned in [1] "surprisingly, there does not seem to be an extensive research literature on the use of scientific calculators for learning mathematical concepts" and we hope that our experience will be helpful for teachers of higher school and lecturers of colleges and universities.*

### 1. Introduction

The permanent discussion about using Calculators for teaching and learning Mathematics commented on in Kissane and Kemp, (2013), must be now reversed from "whether the use of calculators is a positive addition to the mathematics classroom" to "what effective practices do calculators entail" and "new lines of research should begin to explore the conditions, resources, and contexts needed to maximize the degree to which calculator can enhance the teaching and learning of mathematics". The work of authors Kissane and Kemp, (2013a;2013b) give a wide systematic set of exercises for calculators users. Our aim is to present some non-standard problems and other, interesting from educational point of view, aspects connected to the use calculator in teaching Calculus.

At a present time, apart from many kinds of existing calculators (graphical, programmable, symbolical) their functions may be fulfilled also by various applications for Note-books, Tablets or Smartphone widely used by students in the mathematics classes and homework. That results in decreasing use of more developed and considerable more expensive calculators such as symbolical or graphical ones. The main reason for use of "simple" scientific Calculators are restrictions imposed for kinds of Calculators allowed on exams. Despite it, all students of our engineering College use only fx-991ES PLUS CASIO Scientific Calculator, that is a none-expensive and very

comfortable computation device. For this reason we will illustrate our principles and ideas of using Calculators in teaching Calculus on fx-991ES PLUS CASIO machine.

## **2. Lecturer and student's acquaintance with calculator.**

As a matter of fact, most lecturers in our college (and as well as in the other higher educational school in our country) haven't used such standard modern calculator as fx-991ES PLUS and instead they use the previous generations of calculators which were in their possession from the time of their studies. Most of lecturers are from the "pen and paper" generations and all their calculations were done without powerful devices. They think that it is the best way for contemporary students as well. But it is impossible to stop technological progress; we must to take it into account instead of neglecting to use newly widespread computation devices for the benefit of student's comprehension of the main ideas and algorithms of the calculus course.

The student's attitude to calculators as a convenient device for computations will not change without lecturers' impact. The students need proper instructions from the teachers for effective use of advanced options of the modern scientific calculator.

## **3. Advantages and Disadvantages of Calculators**

We share the opinion of negative influence of calculators using in primary school because arithmetic's operations are vitally important for development of logical thinking, memory and important learning habits of young pupils (brain formation). On the contrary, using of calculators in high education makes it possible to concentrate more interest on thinking about problems which need concept understanding and creative approaches. But success in this direction firmly depends on the way of using calculator in classroom and instructions given by the teacher.

We are interpreting calculator as an interactive mathematical dictionary and, in Calculus Course, as a powerful factory of elementary computation and investigation of functions.

## **4. Investigation of calculator**

We try to attain the student's attention to some problems connected to calculator's features, such as:

What is speed of calculator's computations?

What algorithms are using in calculator for some computations?

What are the restrictions and possible mistakes of calculator?

How can we check the results of symbolic calculations by calculator use?

Is it possible for the pupil to win some computation competitions with calculator?

## **5. Investigation of speed of calculators computations.**

Most students don't think about the speed of calculator because they see that calculator did required computations almost instantly after data input. We ask the question how you can know, approximately, the speed of its computation? Not many students may answer these questions – all calculations they do are done by calculator instantly. A following

hint is suggested: Think about the operation  $\sum$  and ask Casio to calculate, for example,

the sum  $\sum_{x=1}^{100} (x^3)$ .

Here some experimental examples

$\sum_{x=1}^A (0)$  calculate for A=100, 1000, 10000 and check by stopwatch time of computation

	A=100	A=1000	A=10000	A=100000
$\sum_{x=1}^A (0)$	“instantly”	7.5 sec	72 sec	12 minutes
$\sum_{x=1}^A (1)$	“instantly”	7.5 sec	72 sec	
$\sum_{x=1}^A (X)$	“instantly”	9 sec	85 sec	
$\sum_{x=1}^A (X^3)$	2 sec	22 sec	3 min 45 sec	

And as to know how many calculation were done to some fix moment of calculation?

For example try to stop calculation of  $\sum_{x=1}^{100000} (1)$  after 10 , 20 , 30, 60.... seconds by

pressing the ON key and check the value of X by pressing ALPHA , X, =

The values of X will be about as the next's: X(20)=2944, X (30) =4353, X(60)=8533

Try it for the sums  $\sum_{x=1}^{100000} (0)$ ;  $\sum_{x=1}^{100000} (x)$ ;  $\sum_{x=1}^{100000} (x^2)$ ;  $\sum_{x=1}^{100000} (x^3)$ ; and think about the differences of received results.

At the first Calculus lecture we need the sum  $1+2+3+\dots+100$  (and after that the sum  $1^3 + 2^3 + 3^3 + \dots + 100^3$ ) calculations. We ask the students to do it on their calculators, but we ask them before the start of computations to estimate the time needed for this calculations. This is not difficult but non-trivial problem for the students.

We ask “what is probable time to input of the data to calculator?”

As a rule none of the students give the correct answer and we start discussion about this problem.

The first question is what elementary operation did we do with calculator?

- **key pressing**- think that one “**press key**” operation needs about half of a second (and we ask to check this suggestion) So for time estimation we need to know the number of “key press” operations for this sum computation.

At first for input of the numbers from 1 to 100 we will need:

For one-digit numbers (from 1 to 9) 9 “press-key” operations;

For two-digit numbers (from 10 to 99)  $90 \times 2 = 180$  “press-key” operations;

For three-digit number (there for number 100 only) 3 “press-key” operations;

For 99 “+” operations we need 99 key pressing

At the end for “=” we need 1 key pressing

So for the straight computation of the given sum we will need  $9 \times 1 + 90 \times 2 + 99 + 1 = 289$  (really more because memory restrictions of calculator we need addition “exe” operation – as we check only 36 first numbers may be input with “+” before “=” key pressing)

Key pressing operations and if we think that the single key operation needs half of a second, we will need about 150 seconds or 2.5 minutes for the sum calculation. We ask the students to check at home our suggestion using this way “straight” computation of the given sum. We ask who did this straight calculation without mistake and how to know that the executed calculation was really correct. As a matter of fact not many of students get the correct result. Note that if pressing of calculators key for total summation were done without looking at calculators display the answer was always wrong – the cause is the restriction of memory and after some number of key pressing all next pressing do nothing- check it for example for the sum  $1+1+1+1+1+\dots$  - the maximum number of terms will be 50 and not more (you may press 100 times, or without knowing number of times (only more than 50)- the result was the same (5). For the sum  $100+100+100+100+\dots+$  the maximum number of terms will be 25.

But pressing of calculator’s key with checking the results on the screen takes more than half of second and so our approximate time needs some correction.

We ask the students give approximate time for direct calculation of the sum

$$1^3 + 2^3 + 3^3 + \dots + 100^3$$

For this sum input an additional operations to needed for the sum  $1+2+3+\dots+100$  are after all number press Key  $x^{[1]}$  (1) after that press Key 3 (1) and after that press the central Key for return to addition operation. These operations take about 300 additional pressing and about 150 seconds or 2.5 additional minutes for input and total about 5 minutes. We ask to check it by direct input of the sum  $1^3 + 2^3 + 3^3 + \dots + 100^3$  in calculator (use stopwatch). In fact it need more than 5 minutes.

We ask the students how this long step summation can be quickly and correctly done by their fx-991ES PLUS Calculator and discuss about  $\sum$  operator which have this

Calculator. By use of this operations the sum  $1+2+3+\dots+100$  may be calculate

instantly: 
$$\sum_{x=1}^{100} x = 5050$$

The students can see that it is an excellent option but what about the sums of big amount of terms. For example  $\sum_{x=1}^{1000} x$  it takes as 20 seconds, and  $\sum_{x=1}^{10000} x$  (85 sec) or  $\sum_{x=1}^{20000} x$  (about 3 minutes) and what about the sum  $\sum_{x=1}^{100000} x$ ? (about 14 minutes) and the result 5000050000.

It was the first example from which the students have seen that the calculator not always may did calculations instantly.

Other question is: May we sometimes win with a Calculator in computation process?

We ask the students how this long step summation can be quickly and correctly done by the use of the suitable precise or approximate formula (if exist) in order to demonstrate the great power of symbolic calculations.

## 6. Limit problems.

Proper use of “Casio” options gives to students good possibilities for understanding of such main calculus notions as limit, derivative, integral. We demonstrate how to check and to understand limit notion by calculator and what are the restrictions of calculator.

For example:

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$$

For the series of substitutions:  $x = 10, x = 100, x = 1000, x = 10^{10}, x = 10^{12}, x = 10^{13}$  by use of Calc option, we can see the stabilization of outputs:

2.59374246, 2.704813829, 2.716923932, 2.718281828, 2.718281828, 2.718281828

But as a result of substitutions:  $10^{14}$  the calculators output change suddenly to 1.

What is the reason? The restrictions of calculator which in its calculations of the base  $1 + 10^{-14}$  saves it as 1 and after that computation each power of 1 is 1. So we recommend not to exaggerate in our advance in the process  $x \rightarrow a$ , or  $x \rightarrow \infty$

Other example:  $\lim_{x \rightarrow 0} (\cos x)^{1/x^2}$

For the series of substitutions:  $x = 0.1, x = 0.01, x = 0.001, x = 10^{-4}, x = 10^{-5}, x = 10^{-7}$  by use of Calc option, we can see the stabilization of outputs:

0.6060240772, 0.6065256052, 0.6065306112, 0.606530659, 0.6065306597, 0.6059244322, 1

We can see also in this example that calculator give non-correct answer already for  $x = 10^{-6}$  and change the result dramatically for  $x = 10^{-7}$

We suggest to students: Be careful in the advance of the variable and not give to its value to be too close to “the end” of the limitation process.

## 7. The derivative of a function

We ask the students to check the way how calculator computes the value of the derivative.

Leibniz's notation  $\frac{dy}{dx}$  for derivative and vertical line for substitution of x value

$\frac{d}{dx} \left[ \right]_{x=a}$  hint at the numerical derivative computation by means of the difference quotient of (the change in y divided by the change in x)

In symbolic calculus we obtain  $\left. \frac{df(x)}{dx} \right|_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$  and in numerical

computation calculator take some value of h, close to 0 and obtain  $\left. \frac{df(x)}{dx} \right|_{x=a}$  as

$$\frac{f(a+d) - f(a)}{d}$$

What value of h can the calculator take (and is it constant for all computation of derivatives)?

Some experiments for this question need to be done.

For example:

For experimental computation of the derivative of  $f(x) = x^3$  at the point 2 (precise value is  $3x^2|_2 = 12$ ) we may input the function

$$\frac{(2+D)^3 - 2^3}{D}$$

and then press the CALC key. After that the several values for D may be

given:

$$D = 0.1 \rightarrow 12.61; D = 0.01 \rightarrow 12.0601; D = 0.001 \rightarrow 12.006001; D = 10^{-6} \rightarrow 12.000006$$

$$D = 10^{-6} \rightarrow 12.000006; D = 10^{-7} \rightarrow 12;$$

For experimental computation of the derivative of  $f(x) = e^x$  at the point 0 (precise value is  $e^x|_0 = 1$ ) we may input the function (expression)

$$\frac{e^D - 1}{D}$$

and press CALC key. After that the several values for D may be given:

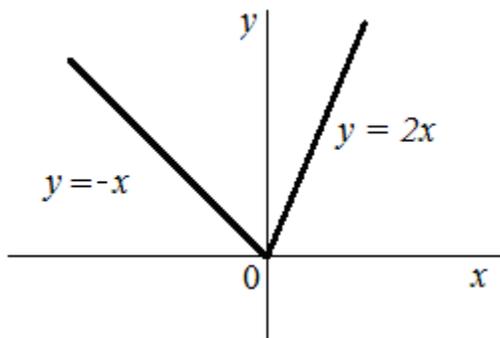
$$D = 1.051709181 \rightarrow 12.61; D = 0.01 \rightarrow 1.005016708; D = 0.001 \rightarrow 1.000500167$$

$$D = 10^{-5} \rightarrow 1.000005; D = 10^{-6} \rightarrow 1.0000005; D = 10^{-7} \rightarrow 1;$$

The answers of calculator may be incorrect at singular points where the derivative doesn't exist. For example  $f(x) = |x|$  (Abs key). The calculator gives  $\left. \frac{d}{dx} (|x|) \right|_{x=0} = 0$ . But

the correct answer – the derivative doesn't exist at point 0.

Now try the function with graph



The equation for this function is  $1.5|x| + 0.5x$  and calculator gives

$$\frac{d}{dx}(1.5|x| + 0.5x) \Big|_{x=0} = 1$$

The investigate question is how were calculate these incorrect answers by calculator?  
But the correct answer – the derivative doesn't exist at point 0.

**From these incorrect answers we may think that calculator's schema for computations may be**

$$\frac{f(a + D) - f(a - D)}{2D}$$

Where D is very small positive number chosen by calculators program. But this option not gives the results which calculator obtains in mention above examples...

### 8. Understanding of approximations formulas

Think about calculation the value of the function  $\frac{x^4 + 20x^3 - 50x^2 + 240}{x^3}$  for  $x=1000$

Without thinking before calculations, most students takes their calculators and do the direct formula computation

Some options for this calculation exist for Casio fx 99ES users

Numerical input only

$\frac{1000^4 + 20 \cdot 1000^3 - 50 \cdot 1000^2 + 240}{1000^3}$  needs about 1 minutes and gives the result 1019.95

Use of symbolic input of expression and CALC Key needs about 1 minutes too but now

we can instant calculate the values of the function  $\frac{x^4 + 20x^3 - 50x^2 + 240}{x^3}$  for any  $x$  (that

is in calculator's numbers range).

The question for students is "Is it possible to do such calculations for the big values of  $x$  without calculator use and as a matter of fact more quickly than calculator?" Let's do some competition with calculator.

The aim of such discussion is an introduction to the asymptotic behavior of functions and understanding of usefulness of the presentation of given function in equivalent form:

$$x + 20 - \frac{50}{x} + \frac{240}{x^3}$$

From this presentation must be clear for students that for big values of  $x$  the value of the function is close to  $x + 20$  and as more big is  $x$  – more precise is this simple calculation and we ask the students to compare the approximate calculation  $x+2$  with calculators precise results.

### 9. Definite Integral computations.

What are methods used by Casio fx for computation?

We can see that for  $f(x) = x^2$  the calculator give the result  $\int_0^{1000} x^2 dx$  instantly

For the function  $f(x) = \sin x$  the calculator give the result  $\int_0^{1000} \sin x dx$  after about 12

minutes and the result is  $\int_0^{1000} \sin x dx = 0.43762092370$ .

Check it by use of symbolic calculus formula

$$\int_0^{1000} \sin x dx = -\cos(x)_0^{1000} = 1 - \cos 1000 = 0.4376209237$$

Here we can see how precise numerical computations were (in spite that the process was so lengthy)!

Try did integral computation by trapeze formula

### 10. Conclusions.

Our experience show that mention above approaches are interesting and useful for most of the students and activate their research abilities. The knowing of calculator's features is valuable not for Calculus teaching but to others disciplines too.

## REFERENCES

- Burrill, G. et al (2002) Handheld graphing technology in secondary mathematics: Research findings and implications for classroom practice. *Michigan State University: Texas Instruments*.
- Drijvers, P. & Weigand, H.-G. (2010) The role of handheld technology in the mathematics classroom. *ZDM Mathematics Education*, 42, 665-666.
- Idris, N. (2005) *Teaching and Learning of Mathematics: Making sense and developing cognitive abilities*. Kuala Lumpur: Utusan Publications.
- Ronau, R., Rakes, C., Bush, S., Driskell, S., Niess, M. & Pugalee, D. (2011). NCTM Research Brief: Using calculators for learning and teaching mathematics. Retrieved 29 July 2012 from <http://www.nctm.org/news/content.aspx?id=31192>
- Kissane and Kemp (2013a). Conceptual development and the modern scientific calculator: Using a forgotten technology
- Kissane and Kemp (2013b) *Learning Mathematics with ES PLUS Series Scientific Calculator*. Cassio Worldwide Education Website
- Kissane and Kemp (2012). Calculators and the mathematics curriculum. In W.-C. Yang, The place of calculators in mathematics education in developing countries. *Journal of Science and Mathematics Education in Southeast Asia*. 35(2), 102-118.
- Kissane, B. & Kemp, M. and Nyaumwe, L. (2006) Investigating Zimbabwean mathematics teachers' dispositions on the 'O' Level calculator syllabus 4028. *South African Journal of Education*, 26(1), 39-47.
- M. Majewski, T. de Alwis & K. Khairiree (Eds.) *Creative and Critical Thinking in Mathematics Through Technology: Proceedings of the 17th Annual Conference of the Asian Technology Conference on Mathematics*. (pp 178-187) Bangkok, Thailand: ATCM Inc.
- Odera, F. & Ochanda, O. (2011) Challenges in the use of scientific calculators in the teaching and learning mathematics in secondary school education. *International Journal of Information and Communication Technology Research*, 1(5), 230-238.
- Peirce, R., Stacey, K., & Wander, R. (2010) Explaining the didactic contract when handheld technology is permitted in the mathematics classroom. *ZDM Mathematics Education*, 42, 683-695.
- Ronau, R., Rakes, C., Bush, S., Driskell, S., Niess, M. & Pugalee, D. (2011). NCTM Research Brief: Using calculators for learning and teaching mathematics. Retrieved 29 July 2012 from <http://www.nctm.org/news/content.aspx?id=31192>
- Ruthven, K. (1996) Calculators in the mathematics curriculum: The scope of personal computational technology. In Laborde, C. (ed.) *International handbook of mathematics education*. Dordrecht: Kluwer.