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Logic and Inequalities: A Remedial Course Bridging Secondary School and Undergraduate Mathematics

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This paper is an informal discussion of a remedial lecture course that I plan to teach in the 2015–16 academic year in the Foundation Studies programme in my university. The course is set at the level bridging GCSE Mathematics as it is taught to students up to the age of 16 in secondary schools in England, and undergraduate mathematics courses for ages 18+. It does not overlap A Level Mathematics and Further Mathematics as they are taught in England for ages 16 to 19.

Introduction

This paper is an informal discussion of a lecture course that, subject to approval, I plan to teach next academic year in the Foundation Studies programme in my university. The paper expands its earlier version [1]. The proposed course is an update of the course that I taught every autumn since 2003.

In England, Foundation Studies courses in mathematics are intermediate Year Zero courses offered to students who were conditionally accepted to study at the university (mostly in Engineering and other STEM degree programmes), but who had not studied, or had not got good grades in, A Level Mathematics, or to students from overseas whose secondary school diplomas are not recognised in England. My particular course is large: 350 students, about 80 of them are from overseas.

My course serves as an introduction to the language of mathematics (well, to its English dialect), and to mathematical thinking.

Outline of the course

The course comprises 22 lectures of 50 minutes each, twice a week over 11 weeks, plus one tutorial class a week.

Sets (6 lectures). Sets and their elements, equality of sets. The empty set and its uniqueness. Finite and infinite sets. ***N***, ***Z***, ***Q***, ***R***. Subsets: union, intersection and complement. Venn diagrams. De Morgan's Laws. Boolean algebra of sets.

Logic (8 lectures).
Propositional Logic. Statements and connectives. Conjunction, disjunction, negation, conditional, their interpretation in human languages. Truth tables. Material implication. Contrapositive and converse to conditional statements. Logical equivalence and tautologies. Boolean Algebra of Propositional Logic.
Predicate Logic. Predicates and relations. Universal and existential quantifiers. Some basic logic equivalences of Predicate Logic.

Proof. Proof of statements of the form $(\forall x)p(x)$ and $(\exists x)p(x)$ Proof of conditional statements

$$(\forall x)(p(x) \rightarrow q(x)).$$

Contrapositive and converse. Proof by contradiction. Proof by induction and computation by recursion.

Inequalities (8 lectures). Inequalities. Solution of inequalities containing unknown variables. Linear inequalities with one or two variables, systems of linear inequalities with two variables. Graphic representation of the solution sets of inequalities. Some simple problems of linear optimisation in two variables. Quadratic inequalities with one variable. Methods of proof for inequalities.

Backgrounds and justification

Why Logic? A very pragmatic justification

The new course contains almost everything necessary for mathematically competent writing and handling macroses for time-dependent EXCEL spreadsheets—and not much else. Time-dependent spreadsheets are the daily bread of practical computing in engineering and business; they are the principal mathematical tool of project management.

A reasonable grasp of propositional logic is useful for learning programming languages and for design and use of digital circuits—skills which are necessary for almost all engineering disciplines.

Inequalities are crucially important for applications of mathematics

To give just a few examples,

- Anything which contains the word "estimate"' in its name, whether it is in Engineering or in Economics, is based on inequalities.
- Anything which contains the word "approximation" in its name, whether it is in Engineering or in Economics, is based on inequalities.
- Anything which contains the word "optimisation" in its name, whether it is in Engineering or in Economics, is based on inequalities.

In addition,

- Inequalities are mathematical tools for control of errors in measurement and in experimental data, as well as for handling rounding errors in computations.
- Statistics is all about inequalities.
- Perhaps most importantly, the instinctive feel of inequalities makes the basis of quick "back-of-the-envelope" estimates and "guesstimates," the essential part of engineering thinking.

In words of Bertrand Russell,

> *Although this may seem a paradox, all exact science is dominated by the idea of approximation. When a man tells you that he knows the exact truth about anything, you are safe in inferring that he is an inexact man. Every careful measurement in science is always given with the probable error ... every observer admits that he is likely wrong, and* ***knows about how much wrong he is likely to be***. [Emphasis is mine. – AB]

Inequalities are badly taught at school

Markov's Inequality:

If X is any nonnegative random variable and $a > 0$ then $\mathbf{P}(X \geq a) \leq \mathbf{E}(X) / a$, the first but fundamental result of the theory of random variables—and the basis of the entire Statistics—is no more than a primary school level observation about inequalities and can be formulated as an arithmetic "word problem" about anglers and fish:

> 50 anglers caught on average 4 fish each. Prove that the number of anglers who caught 20 or more fish each is at most 10.

A solution is simple. Assume that there were more than 10 anglers who caught at least 20 fish each; then these 10 anglers caught together more than $20 \times 10 = 200$ fish—a contradiction.

Unfortunately, we cannot expect that all students entering English universities are able to produce this argument. There are two reasons for that:

- This is a proof from contradiction—and this is why basic proofs from contradiction are part of the course.
- The argument requires simultaneous handling of two types of inequalities, "x is more than y," denoted $x > y$, and "x is at least y," denoted $x \geq y$.

Alas, I many times met students who were asking me questions of that kind:

> *How can we claim that* $3 \geq 2$ if *we already know that* $3 > 2$?

This fallacy is a symptom of a dangerous condition—logical deficiency. Handling inequalities demands stronger logical skills than mechanical manipulation of equations.

Moreover, inequalities are frequently more important than equations! For example, besides the equation for a straight line in the plane,

$$ax + by - c = 0,$$

closely related inequalities

$$ax + by < c, \quad ax + by \leq c, \quad ax + by \geq c, \quad ax + by > c$$

are no less important: they describe the way the line cuts the plane in two halves (and therefore have natural applications, say, in computer graphics). To give just one example, here is a simple problem:

> Answer without sketching graphs: do points (1,3) and (-2, 4) lie to the same side off the line $2x + 3y - 1 = 0$ or belong to the opposite sides?

Mathematical logic allows us to see connections between inequalities and equations which play an important role in many practical problems.

The natural affinity of the theory of inequalities and elementary logic

Inequalities fit happily into the course which starts with sets and logic not only because they need logic, but also because, in a way of reciprocity, systems of simultaneous inequalities provide accessible material for learning and applying techniques of logic, deduction and proof. Basic Boolean Logic: conjunction, disjunction, negation, comes into play very naturally. A system of two simultaneous inequalities is the *conjunction* of inequalities, the solution set of the system is *the intersection* of the solution sets of individual inequalities.

The inequality $x^2 > 1$ is *equivalent to the disjunction* of inequalities $x > 1$ and $x < -1$

$$(\forall x)(x^2 > 1 \leftrightarrow (x > 1 \wedge x < -1)),$$

and the solution set of $x^2 > 1$ is *the union* of the solution sets of $x > 1$ and $x < -1$:

$$\{ x \mid x^2 > 1 \} = \{ x \mid x > 1 \} \cup \{ x \mid x < -1 \}$$

The *negation* of the inequality $x^2 > 1$ is $x^2 \leq 1$, and the equation $x^2 = 1$ is the *conjunction* of inequalities $x^2 \geq 1$ and $x^2 \leq 1$:

$$(\forall x)(x^2 = 1 \leftrightarrow (x^2 \geq 1 \wedge x^2 \leq 1)).$$

Even more remarkable (and the reason why quadratic inequalities need to be discussed not only for their practical importance, but also as an illustrative material for logic), that the inequality $x > 1$ *implies* the inequality $x^2 > 1$,

$$(\forall x)(x > 1 \rightarrow x^2 > 1),$$

but $x^2 > 1$ *does not imply* $x > 1$,

$$\neg(\forall x)(x^2 > 1 \rightarrow \mathrm{x} > 1),$$

or, rewriting this statement in a logically equivalent way,

$$(\exists x)\ ((x^2 > 1) \wedge \neg(x > 1));$$

in plain language, it means

there exists x such that $x^2 > 1$ but $x \leq 1$.

Systems of simultaneous inequalities are *predicates* – *unary*, in case of systems of inequalities in one variable, and *binary* – if we have two variables.

Is Logic too hard?

At an elementary restricted level – no, it is not. Logical formulae that I gave as examples might appear to be excessively complex. But the logical connectives $\neg$ (negation), $\wedge$(conjunction), and $\vee$(discjunction) are routine operators in computer coding. The *universal quantifier* $\forall$ and the *existential quantifier* $\exists$, if used sparingly, help students to develop sharper reasoning skills. In an example above, the statement

it is not true that $x^2 > 1$ implies $x > 1$

translates into symbolic notation as

$$\neg(\forall x)(x^2 > 1 \rightarrow x > 1),$$

and is logically equivalent to

$$(\exists x)((x^2 > 1) \wedge \neg(x > 1)),$$

which, in plain language, means

there is x such that $x^2 > 1$ but $x \leq 1$.

Without prior exposition to basic symbolic logic, many students would have difficulties in understanding that the statement

(A) it is not true that $x^2 > 1$ implies $x > 1$

is the same as

(B) there is x such that $x^2 > 1$ but $x \leq 1$.

My aim, of course, is to help my students to eventually see that (A) and (B) are the same without resorting to logical symbolism. Graphic representation of inequalities (and therefore some basic set-theoretic thinking) is a useful stepping-stone: both (A) and (B) are equivalent to saying that

the set of solutions of $x^2 > 1$ is not a subset of the set of solutions of $x > 1$.

I will restrict the use of alternating quantifiers to gently introduced single change cases, $\forall\exists$ and $\exists\forall$. Indeed, the $\forall\exists$ combination already has to be handled with great care, it triggers the explosion of infinity:

for *every* number *there is* a bigger number,

$$(\forall x)(\exists y)(x < y).$$

I will definitely avoid the notorious $\forall\exists\forall$, the perilous stumbling block of the ε-δ language of the real analysis.

Some even more general methodological observations

A bit of cognitive science and didactic considerations

An additional link between inequalities and logic is provided in example that I am systematically using in the logic part of the course: (pre-)order relations and their logical combinations appearing as kinship relations in human societies. If $F(x,y)$ denotes "x is the father of y", the difference between

$$(\forall y)(\exists y)F(\mathrm{x,y})$$

and

$$(\exists x)(\forall y)F(x,y)$$

becomes instantly self-evident. Kinship is remarkably self-evident, and for very deep cognitive and evolutionary reasons—already apes and even monkeys have sophisticated kinship systems.

The wonderful book *Baboon Metaphysics* [2] provides some astonishing evidence—and please notice that the book contains a formal definition of strict linear order:

> *The number of adult males in a baboon group at any given time ranges widely, from as few as 3 to as many as 12. Regardless of their number,*

> *however, the males invariably form a linear, transitive dominance hierarchy based on the outcome of aggressive interaction (a linear, transitive hierarchy is one in which individuals A, B, C and D can be arranged in linear order with no reversal that violate the rule 'if A dominates B and B* dominates *C, then A dominates C'). Although the male dominance hierarchy is linear, transitive, and unambiguous over short periods of time, rank changes occur often (Kitchen et al. 2003b), and a male's tenure in the alpha position seldom lasts for more than a year.* [p. 51]

> *Like males, female baboons form linear, transitive dominance hierarchies. There, however, the similarity ends. Whereas male dominance ranks are acquired through aggressive challenges and change often, female ranks are inherited from their mothers and remain stable for years at time. Furthermore, most female dominance interactions are very subtle. Although threats and fights do occur, they are far less common and violent than fights among males. Instead, most female dominance interactions take the form of supplants: one female simply approaches another and the latter cedes her sitting position, grooming partner, or food. The direction of supplants and aggression—and the resulting female dominance hierarchy—is highly predictable and invariant. The alpha female supplants all others, the second-ranking supplants all but the alpha, and so on down the line to the 24th- or 25th-ranking female, who supplants no one.* [p. 65]

It is truly remarkable to what degree the concept of linear order is self-evident to humans. But anyone who taught freshmen knows that the concept of equivalence relation is considerably harder. The reason is that transitivity of dominance is obvious at the level of the monkey bits of our brains. The transitivity of equality is a much later, in evolutionary terms, social construct. In the powerful scene in the film *Lincoln* [3], Abraham Lincoln says to his astonished aids:

> *Euclid's first common notion is this: Things which are equal to the same thing are equal to each other. That's a rule of mathematical reasoning. It's true because it works. Has done and always will do. In his book, Euclid says this is* 'self-evident.' *You see, there it is, even in that 2,000-year-old book of mechanical law. It is a self-evident truth that things which are equal to the same thing are equal to each other.*

The scene is a fiction, but it is very true in spirit to a number of well-documented quotes from Lincoln where he uses references to Euclid as a logical and rhetoric device (see, for example, a real quote from him later in this paper).

I refer to Abraham Lincoln as to authority to prove that the level of abstraction in the course is set carefully and balanced by self-evident nature of principal examples.

Cultural and linguistic issues

Students in the course come from a variety of socioeconomic, cultural, educational and linguistic backgrounds. Just at a level of basic notation, I have to deal with students who, in their school mathematics, were using two different symbols for multiplication:

$$2 \cdot 3 = 6 \quad \text{and} \quad 2 \times 3 = 6,$$

and three different symbols for division:

$$6/3 = 2; \quad 6 : 3 = 2; \quad 6 \div 3 = 2.$$

Some countries use decimal point:

$$\pi = 3.1415\ldots,$$

while others prefer decimal comma:

$$\pi = 3{,}1415\ldots,$$

this list can be easily continued.

Even more obstructive are invisible differences in the logical structure of my students' mother tongues. For example, the connective "or" is strictly exclusive in Chinese: "one or another but not both", while in English and Russian "or" is mostly inclusive: "one or another or perhaps both", with an occasional slip into the exclusive mode: "her *or* me!" Meanwhile, in mathematics "or" is always inclusive and corresponds to the expression "and/or" of the bureaucratic slang. But this is only the beginning: in Croatian, for example, there are two connectives "and": one *parallel*, to link verbs for actions executed simultaneously, and another *consecutive* (and some signs of that can be detected in Russian).

Articles are a special issue; many languages have no definite articles, and therefore, in the very first lecture, some time is devoted to the detailed discussion of the expression

the empty set.

The list can be continued almost indefinitely; the key point here is that teaching elementary logic to a ethnically and linguistically diverse audience inevitably becomes an exercise in multiculturalism, and a lecturer has to be prepared to face the challenge.

Material implication

The principle of *material implication* states:

it is true that

- "false" implies "true", or, in symbolic notation,

$$F \to T,$$

and

- "false" implies "false", which becomes, in symbolic notation,

$$F \to F.$$

Material implication is the hardest part of cultural accommodation of my students. This principle is unacceptable and even morally offensive to many of them, and for deeply rooted cultural reasons; their objections have to be treated with great respect.

Indeed, the principle of material implication follows from the expectation that the statement

$$P \to (Q \to P)$$

is always true, regardless of the validity of P or Q. The latter has meaning

If P is true, it is true regardless.

There is a Russian proverbial saying about card games:

Ace is an Ace, even in Africa,

and a more general Hungarian one:

A gentleman is a gentleman, even in Hell.

The glorious tautology

$$P \to (Q \to P)$$

is the proclamation of independence of thinking:

> if something is true, it is true regardless of what others say about it;

and, after specifying the meaning of the word "others", becomes anti-authoritarian and subversive:

> if something is true it is true regardless of what the (traditional and absolute) authority says about it.

Teaching logic means teaching thinking, and, as soon as you start teaching thinking, you get into the heat of dead serious ideological debates. A lecturer has to tread this path very, very cautiously, taking great care with every his/her step.

Psychological support and the ``radical remediation''

By the nature of the course, some of my students had unhappy experiences with school mathematics. As a lecturer, I have to give them moral and psychological support. It helps that the content of the course deviates from the standard school curriculum—this allows me tell my students, that they are given a chance to start from clean page and leave their fears and frustrations behind. Using the words coined by Roman Kossak, it is a "radical remediation" of mathematical learning.

Finally, a bit of metamathematics

As the great mathematician and philosopher Alfred Tarsky had explained in his seminal work [4] on elementary geometry,

$$\text{Logic} \cup \text{(quadratic and linear inequalities over the reals)} \subseteq \textit{Euclid's Elements}.$$

The course described in this paper, is, from the formal metamathematical point of view, the Euclidean Geometry in (unrecognisable for a lay person) disguise smuggled back into the secondary school level curriculum.

Therefore the material in the course could be made as rich as the Euclidean Geometry, with all the opportunities for rigorous definitions and rigorous (and not very complicated) proofs. The course can be slightly extended and reinforced to arm the student with the same power of deduction as was demonstrated by Abraham Lincoln (and this time these are his own words):

> *One would start with confidence that he could convince any sane child that the simpler propositions of Euclid are true; but, nevertheless, he would fail, utterly, with one who should deny the definitions and axioms. The principles of Jefferson are the definitions and axioms of free society. And yet they are denied, and evaded, with no small show of success. One dashingly calls them 'glittering generalities'; another bluntly calls them 'self-evident lies'; and still others insidiously argue that they apply only 'to superior races'.* [5]

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Disclaimer

The author writes in his personal capacity and the views expressed do not necessarily represent position of his employer or any other person, organisation or institution.

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[2] D. Cheney and R. Seyfarth, *Baboon Metaphysics: The Evolution of a Social Mind*. University of Chicago Press, 2008. ISBN-10: 0226102440. ISBN-13: 978-0226102443.

[3] *Lincoln*, http://www.thelincolnmovie.com. Director: Steven Spielberg; in the title role: Daniel Day-Lewis; screenplay: Tony Kushner.

[4] A. Tarsky, *A Decision Method for Elementary Algebra and Geometry*, prepared for publication by J. C. C. McKinsey, U.S. Air Force Project RAND, R-109, the RAND Corporation, Santa Monica, 1948, iv+60 pp.; a second, revised edition was published by the University of California Press, Berkeley and Los Angeles, CA, 1951, iii+63 pp.

[5] A. Lincoln, *Collected Works*, 3:375, quoted at C. S. Morrissey, Spielberg's Lincoln: Politics as Mathematics, *The Catholic World Report*, December 19, 2012, http://www.catholicworldreport.com/Item/1822/spielbergs_ilincolni_politics_as_mathematics.aspx; Morrissey, in his turn, quoted G. Havers, *Lincoln and the Politics of Christian Love*. Columbia, Missouri: University of Missouri Press, 2009, p. 72.