MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
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Editorial: *Building Bridges*

This, 23rd issue of MTRJ presents a bridge of Mathematics Education starting in New Zealand, via Odessa in Ukraine through Great Britain and anchoring final in NYC in US. The bridge starts very gently with wisdom of Steve Arnolds' recognition of a deep contemporary contradiction between our mathematical conceptions and the human world. He asks for the a fundamental change in the nature of our thinking mathematics so that it is at one with contemporary reality of our human world. This quest for unity is reflected also in the very concept of Teaching-Research facilitated by MTRJ whose ultimate goal so succinctly expressed by Steenhouse as the creation of the classroom methodology through "acts which are at once educational act and a research act". The question is how to do it. For us, the hint along that heroic pathway is given by the following realization that " humanity hasn't noticed that we have left behind To Be OR not To Be of Hamlet and have arrived at To Be AND not To Be of the Schroedinger Cat."
Alexandre Borovik's paper pursues similar pathway in search of unity between remedial and advanced mathematics with methods that bring envy to the remedial mathematics instructors who have to conform to mind –dumbing curricula imposed by the central headquarters of the university.
The papers by Klymchuk and Zverova take us into the "bread and butter" zone of our profession that is into the process and the role of mathematical modelling, and interestingly, they also are concerned about connection of mathematics to, this time, real world. What is the effective methodology of that walk back and forth between the mathematics and "real world"? That we might be able to learn from the last paper which anchors the bridge spanning half of the world in the creativity of the Aha moment, which as it turns out from Koestler's theory of bisociation in the Art of Creation (1964) is that bridge we are looking for, so it seems. "Bisociation is the spontaneous leap of insight which connects two planes of thinking which by themselves are unconnected". So it seems that the bridge we've been looking for is in our creativity.

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The limits of a rational mind in an irrational world - the language of mathematics as a potentially destructive discourse in sustainable ecology.

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Keywords:

Sustainability, mathematics, rational, education, language, metaphor, systems ecology

Abstract

The rational language and ideas of mathematics have been created and adapted by humans over millennia. The ideas of number and arithmetic are fundamental to central concepts of civilian life. It becomes easy to forget that mathematics is a metaphor for our understanding of the Universe. Mathematics tell a story about the universe. The language of number and operations allow the human to extend thinking about concepts within the universe. Mathematics is not however, a perfect tool for understanding. It is a good story, but a story nonetheless. It has some very powerful features which support and articulate the human mind that developed the threads of mathematics. It also has some limitations. The collective human rational mind that created mathematics wants to believe so strongly in the predictive power of the story that sometimes the story is more powerful than the irrational reality which it describes. When is mathematics not a good tool? George Orwell's character Winston Smith, in 1984 (Orwell & Prebble, 2007) is forced to accept the absurdity of the notion that 2+2=5 if that is what he is expected to believe.

"You are a slow learner, Winston"

> "How can I help it? How can I help but see what is in front of my eyes? Two and two are four."
>
> "Sometimes, Winston. Sometimes they are five. Sometime they are three. Sometimes they are all of them at once. You must try harder. It is not easy to become sane"

And "Freedom is the freedom to say that two plus two make four. If that is granted, all else follows" To what extent do we accept the status quo because of the established social pressure to do so? Can the irrational world be modeled by mathematics?

Bibliography:

Steven Arnold is a senior lecturer in AUT (Auckland University of Technology) his interest in mathematics education requires him to consider how the ideas of mathematics are passed on from generation to generation. It was in noticing that one consequence of passing on of these ideas, is that we have to give up other ideas. Are the mathematical ideas so robust that they can't be challenged? Starting with an exploration into sustainability in the University, Steven was lead through his teachings to challenge some of the fundamental notions of mathematics, arithmetic and number.

Sustainability

Sustainability is a concept far beyond the conservation of energy and resources. It extends even further than the responsible transmission of effective guardianship of our planet to subsequent generations. The concept of sustainability challenges us to delve into the content and the structures of the acceptable ways of being.

Capra (Fritjof Capra, 2004) reminds us of the systems found within nature as an appropriate basis for our supernatural systems; noting that:

> "It is becoming ever more apparent that our complex industrial systems, both organizational and technological, are the main driving force of global environmental destruction, and thus the main threat to the long-term survival of humanity. To build a sustainable society for our children and future generations, we need to fundamentally redesign many of our technologies and social institutions so as to bridge the wide gap between human design and the ecologically sustainable systems of nature.[1]"

What aspect of the **system** might we consider ready for change?

The United Nations collective wisdom found in the Education Science and Cultural Organisation document articulate that sustainability is emergent thinking.

> "Sustainable development, a constantly evolving concept, is thus the will to improve everyone's quality of life, including that of future generations, by reconciling economic growth, social development and environmental protection." (*UN Decade of Education for Sustainable Development, 2005-2014: the DESD at a glance; 2005 - 141629e.pdf*, 2014)

As sustainability is evolving, changing definitions will be needed. Phrases such as 'improve everyone's quality of life', 'future generations', 'growth' and 'development' have a long established pedigree in education.

The Sustainability Math Home page summarise as follows

- The current state of people is not a morally acceptable endpoint of societal development.
- Humans have reached a state where we are negatively impacting the ability of future generations to meet their needs and aspirations.
- The major types of problems facing humanity have to be addressed simultaneously: there is no ranking of importance.
- The "System" requires fundamental changes. (Sustainability Math Home Page, 2014).

Sustainability depends on concepts of systems thinking; renewable, recyclable and reusable energy flows. Understanding sustainability requires a holistic mind that is aware at once of the local and details level, while simultaneously being globally and dynamically focused. Sustainability encourages an appreciation and celebration of networks, beyond ego-centric to eco-centric. The finite world that we live in needs a different story to the linear, cause and effect and predictable simple dynamics found in our current discourses. We need more than the perceptual world of mathematics to solve problems. The delicate, and finite systems of the global ecology require careful awareness and management, and sustainability thinking needs flexibility and futures-focused problem solving to approach complicated and complex ecologies.

The concept of sustainability is a relatively recent arrival in the educator's lexicon, being popularised as recently as 1965 with the meaning "capable of being continued at a certain level" (Harper, 2001). In this context, sustainability also carries the idea of the judicial and prudent relationship to the consumable components of the finite ecology. Does engagement with the well-known structures of the philosophy of mathematics nurture responsible and ecological modes of thinking?

Maths and Sustainability

When Descartes declared "Cognito, ergo sum"(Watson, 2007) he hailed the rational mind as central to existence, he also joined the worlds of algebra and geometry in his Cartesian plane. Maths, as a model for rationality, was to provide a model for the world. However, we can now see some of the consequences of this world view.

Perhaps the simple, yet powerful, language of maths itself precludes a responsible sustainable world view? Some simple examples may assist in rethinking mathematics and the potential that 'education' has for numbing us to our real world. For example: how can we justify teaching students of the arithmetic concept of "takeaway" when we also live in the real world? When can you takeaway anything? We can't take away rubbish, we only move it out of sight, and even that is just temporary because at the same time we introduce notions of the finite planet as a closed system. What is happening when we say “I have five lollies, and take two away”? Where do the other lollies go? Who gets to eat them?

Through my teaching, research and reflections, there came a realisation that one of the ways in which we process sustainability concepts, are captured within the curriculum ideas themselves. For further clarification, and by example I decided to test the ideas of mathematics, specifically arithmetic and number; in terms of sustainability. My teaching of mathematics forced me to consider carefully some of the components of a mathematic curriculum.

The ideas presented in mathematics do not tell the complete human story. They do not tell the whole story of our planet. The thought occurred to me that the very notions of mathematics, may be representing ideas that inhibit notions of sustainability.

So my questions became "to what extent does the engagement with the well-known structures of the philosophy of mathematics nurture responsible and ecological modes of thinking?" What engagement is needed to reflect critically on the role of maths thinking in sustainability?

Education has many functions, one of which seems to be the transmission of received knowledge. The ideas themselves which have been conveyed from generation to generation mould and shape our ways of thinking. Much has been written on the education reformation and these have centred mainly on politics (who), pedagogy (why), delivery (how), syllabus (when) and assessment (which) practices. What adjustments can be made to the ideas of curriculum content (what)?

Futures thinking offers a new set of tools for engaging with the content of the curriculum. Without delving too much into the emerging structures of complex analyses of education, what emerges in futures thinking is a different approach to education. The 'building blocks' of education; content, knowledge, learning, teaching and assessment morph into new constructs such as languaging, discriminating, co-constructing, third space potential and diagnosis. For now let us just focus on the content of the curriculum.

The study of Mathematics is an interesting curriculum area in that it corners a high profile within the curriculum hierarchy. Ken Robinson (2011) argues for the elimination of the curriculum hierarchy, as there are limitations to the positivist nature of the way ideas are taught within schools. The ideas of a curriculum hierarchy is also challenged by postmodern theorists as Slattery (1997) points out.

Is there something in the mathematical ideas that might inhibit notions of sustainability? Dare we even question the rank of knowledge gleaned by generations? Can we suggest that some ideas – fundamental to our understanding of self – have come to time that they no longer serve all of our needs?

As many authors (F. Capra, 1997; Fisher, 2013; Lovelock, 2000; Orr, 2004; Sahtouris, 1999; *UN Decade of Education for Sustainable Development, 2005-2014: the DESD at a glance; 2005 - 141629e.pdf*, 2014) have done before me, I started to consider sustainability and education in a different way.

Nature

An alternative to using mathematics to control, predict or diminish nature is to use wisdom found from da Vinci. Fritjof Capra's book "Learning from Leonardo: Decoding the Notebooks of a Genius" (2013) shares insight into the mind of Leonardo da Vinci who became aware, 500 years ago, that nature should be a central synthesising agent for the increasingly disparate modes of study. The idea of nature as a guide for education is not new. The word environment is commonly used as synonymous with 'nature'.

The return to 'nature as the guide' is an extremely challenging and subtle reflection and challenge to our received wisdom.

Right and Wrong

The pervasive mathematical framework exists strongly within our education system. An analysis of these ideas starts to challenge some fundamental concepts. The well-loved, and familiar concepts of mathematics themselves started to unravel. The simple, familiar and friendly ideas of the universe that had been 'hard wired' into our brains could be represented *by* mathematics but not represent them entirely. This became disconcerting, disorientating and disturbing. Through traditional schooling methods, we have all been taught mathematics in a way that leant an absolute certainty – mathematics were a set of ideas and they were right. In exploration, though, there is a realisation that some ideas didn't sit right in our world. Mathematics solutions can be simply marked right or wrong, and, so maths maintains a unique place in the school curriculum as having a stark, high stakes, bipolarising quality.

Ideas that students of mathematics have taken for granted ever since their own schooling, as fixed, can become exposed as open for interpretation when the concepts of the apparent absolute, reliable and omnipotent structures of mathematics are explored. The questions raised confront the very nature of mathematics.

In fact the notion of right and wrong, which mathematicians value so much sits at odds with our pluralistic and dynamic understanding of the world. Here we have an entire system of thought based on right (and therefore wrong), which is in itself an unsustainable ideology. From that simple predictable approach to the world of absolutes, where logic commands, and algorithms produce predictable outcomes, comes a way of thinking that is often at odds with our own irrational world and ways of being.

Counter intuition

Many of the different concepts of mathematics are counter intuitive. Imagine the concern when first a child is introduced to finding infinity in a finite space. For example I say to the students "how long will it take for me to jump to the door if each jump halves the remaining distance?"; they realise that halving the distance from here to the door in successive jumps, will last an infinite number of jumps – the door target will never be met. Yet the child's experience tells them that you can just simply touch the door, when you are close 'enough'.

Another example is realising that the number of diagonals in a circle is infinite. Again the concept of infinity challenges us to release the conceptual understanding of the word derived through intuition, sensation and experience and take a leap of faith into reliance on logic.

There is a point where this abdication of trusting the senses becomes perilous ((Abrams, 1996).

Another example helps us to become aware of how much we rely on logic against our own intuition and understanding of the world.

It is because time is continuous and not discrete, that each representation of time is only ever an approximation which leads to the ironic paradox that it is never exactly 2 O'clock. It can be just before two o'clock and just after two o'clock. It can't ever be exactly any given time. Any time is always fractionally before or after the stated time.

Mathematics

Our numbering system is a key to accessing our culture. And numeracy is a highly valued skill and is used as a threshold to higher education, employment, and general social acceptability.

Numbers are a comfortable universal, offered to us from our early days of life. We are surrounded by number as a way of enumerating, and as a comforting cardinal (naming) and ordinal (organising) tool. We have comfort in the existence of numbers knowing them to be 'right' to such an extent that we depend on numbers.

Galileo famously said, "The laws of Nature are written in the language of mathematics." However we realise that this profound statement was while very true, it is not strictly true. There are times when the mathematical understanding of the world breaks down. Now in a time of ecological distress, we need technologies and tools that can match more perfectly our world.

In reality, Mathematics is a highly nuanced poetry that describes the human condition, it mirrors the workings of the human brain (as mathematics is exclusively a product of human thought).

Mathematics tells us our own story, it tells us how the human brain works, and as we strive to make meaning of the world, we do so using the tools available to us; number is one of the ways that we language our experience.

We understand from the progression of numbers, incrementing by one unit at a time, the safety and dynamics of a world that follows simple and immutable rules. Eight is one more than seven. Six is twice three. This comfort derives from endless re-enforcement both from formal mathematics lessons, and from the language developed in the world around us.

There is within mathematics an etiquette of the way we use and apply these rules that is accepted by all, and have to be learned by the un-initiated.

The powerful numbering system is learned early on by young children, and the mastery is often a mark of pride in parents when children learn to recite numbers in order, and later, learn to count (1:1 correspondence).

There is here, a potential challenge to the concept of number itself.

As a founder of mathematical ideas, and an eager sponsor of mathematicians, Plato (Boyer & Merzbach, 2011) aimed to derive a perfect description of the world through mathematics and geometry ("Let none who are ignorant of geometry enter"). Within mathematics there continues to this day an expectation that the simple relationships described in mathematics should be able to neatly describe our complex world. However the real world is not simple, tidy and neat. The real world is full of messiness, unpredictability, human emotion and error. Mathematics describes a predictable world, where error can be eliminated, and it is desirable to simplify and exterminate unwanted complications. Where the two differ, surprisingly it is the human experience in the real world that defers to the all-powerful notions of mathematics.

The simple concept of number only really works in the abstract field of mathematics. Some may choose to argue that numbers are a simple, and neutral tool beyond human value. Numbers, however are far from neutral. Number is a wonderful story device and a great metaphor, it almost works in our real world, but not quite. The same can be said of geometry, statistics, and measurement. Algebra, of course is the purely abstract form of mathematics that exists solely within the conceptual realm.

Numeration

To what extend do the numbers that you accumulate through life, dictate the life you lead. Is someone who gets 97% for physics so much better than the person who got 96.5%? One might get dux, and all sorts of other accolades, the other might drop physics. Equally those who fail one English exam (due to the fact that they speak 4 other languages, and are dealing with the realities of living in a migrant family) may have severely diminished life chances. Does school success (as measured by number in school exam scores) predict life happiness, health, wealth? To what extent are numbers ruling people's choices?

Numbers appear as a powerful predictor of success in our lives, or so we believe. What is the correlation of number to real life? The following example of bananas shows us some limitations of the clean concept of number.

If I have just one banana – it can be eaten as a snack. If I have two bananas, it means that I could substitute them for a light meal. If I have three bananas, there is enough for some friends to share. When I have four bananas I have enough for my immediate needs, and I have got enough for later. With five bananas I have a storage problem.

In mathematics the numbers sit in an orderly line along a number line. Each number bears a strict relationship to others, there are predictable rules for their relative values, and strict applications of how each number operates. 4 is twice 2. 6 is three less than 9. Always.

However in the real world example the strictness of the number systems breaks down: each increment in the number of bananas fundamentally changes the dynamics. It seems there is not a universal mapping of our story of number onto the real world. Even the apparently simple concept of number has more complications than originally thought. Number as quantity is an unreliable metaphor of experience. 5 bananas is not the same as 5 lots of 1 banana.

Quantity and Quality

While counting is one aspect to number (quantitative) there is another aspect (qualitative) of objects which holds another dimension.

If I place an amount of uniform sticks on the table – it is possible to count them. As long as we all agree that for the purposes of the exercise they are similar enough to be grouped together as a homogenous set. Most children become quite comfortable with this 1:1 notion of counting after a while. Piaget (Inhelder & Piaget, 1958) shows it takes some time for the size of the group to be ignored over the 1:1 correspondence; he notes that children will consider there to be more sticks if they are spread further out, even if 1:1 correspondence has already been established.

The child learns to over-ride their intuition and understanding of the real world. Their experience to date is that bigger is more. The child has to let that concept go to embrace the 1:1 correspondence of number to object in the real world. The child learns that the rules of mathematics must be obeyed against the natural sense of experience and natural living.

In another example, if I take some rods of uniform character that are 20, 40, 60, 80 and 100cm long. I could label the first 1, the second 2 and so on. The rods are exactly proportional, in that the 4 and 1 together are exactly the same length as the 5. Where is the '4' ness of the rod? There is only one rod (not 4 individual rods). It is agreed it is '4'. The 4ness is within the quality of the rod, the whole rod is 4; not just a bit of it. In the same way that I am 6 foot tall. All of me is 6 foot tall. It is not that I am made up of 6 lots of 1 foot bits. I have the quality of "6 foot"ness.

Values and Emotions

We put so much faith in numbers, that sometimes we place the power of the digit over the judgement of our experience. This idea of positivism has found a secure home in the teaching of mathematics in schools. We are controlled by numbers, from the early stages of test results, to class position and IQ, to more recently BMI scores, glasses prescriptions, salaries and postcodes. We sometimes forget that numbers are a way to tell the human story. We forget we make them up, not the other way round.

The 16 year old is seen as someone who has more power, is far superior athletically, and generally more cool by the adoring 12 year old cousin. Year 11's have more freedom than year 6's. In many aspects of life it is assumed that it is generally '**better**' to have a bigger number: Salary, IQ, height and so on. In some cases it is **better** to have a lower number: weight, blood pressure, debt. In some aspects of society it is **better** to have a certain set of numbers: postcode, socio-economic status, BMI. The value scale of 'better' exists independently of whether we agree on which quantity is better or not. Number has an emotive and value laden notion. Is it better to be 'smarter' at school? IQ is a very poor predictor of health, wealth or happiness (Noddings, 2003).

The inherent emotive quality attached to some numbers can over-ride other human wisdom. A person may feel fit and healthy, but remain depressed about their weight due to an 'abnormal' BMI. A salary may be sufficient, until it is compared to another. A test score may become a social advancement, rather than a reflection of knowledge. The Chinese have strong identities with the numbers themselves; with the number 8 being strongly linked to prosperity and happiness.

Naming

Numbers are not a neutral label, neither are they particularly consistent in their use as the following examples indicate. When we deconstruct ideas of numeracy further we find some embedded concepts. For example being in the group labelled “4” is not twice as good as being in the group labelled “2”. “Room 17” is not one better than “Room 16”. The use of number as a naming tool (nominal quality) reduces clarity around the relative value of number.

Ordering

In a race there were 5 entrants: The first person had a time of 1:06.05 and the second had a time of 1:10.08 and the third 1:45.93, the fourth person timed 1:50.87 and the fifth 1:50.89. Here number does not indicate anything else than order (Ordinal quality). It does not tell us anymore than the third person came before the fourth and after the second.

Perhaps there are other times when the number, quantity or amount of things fundamentally *changes in kind* rather than a linear *progression of degree* change. The simple numeration model indicating arithmetic relationships starts to collapse.

Numbers do show a relative status, but is -4 degrees twice as cold as -2 degrees? What does that mean?

Is it better to be taller? To what extent do we gear society to an assumed optimum? We find averages, means, medians and mode. Somehow there are advantages or

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disadvantages to being close or far from one of these numbers. It somehow undermines and over-rides the inherent human experience.

Operations

Mathematics is a powerful and central tenet in all of our lives. It provides a fundamental language tool in the way of number and operations. It provides an ordering and thinking tool in the form of logic, algorithms, procedures, and theorems.

Numbers have nothing to do with mathematics; just as letters have nothing to do with literacy. It is not the numbers themselves that tell the whole story of mathematics. Literature is the telling of stories of human endeavour and the richness of culture, and making sense of the world. It is not the letters themselves. The letters are simply the medium. Just as number is a medium in mathematics.

The combination and patterns of letters are a useful tool to assist in decoding the texts but do not in themselves carry any meaning. The same is true for number which is a way of telling the rich human story using the mathematics paradigm; logic, reasoning, pattern, relationships and order.

Letters and numbers tell the story, they allow us to record, relive, and reflect on our culture. The various rules of their combinations and distributions such as for letters; spelling, grammar, punctuation AND in number; operations – addition, subtraction, multiplication and division, squaring etc. are simply tools to tell the story. But what if how they tell the story is not quite right. What if the message is distorted by the medium?

Equals

The central tenet of mathematics is the logical notion of "equals", however what is the place of equals in a world where we celebrate difference? The notion of equality in mathematics derives from logic, but when are two things in reality ever the same? So this equivalence is purely a mathematical construct. No two humans are the same, no two events are the same. No two bananas, and in fact no two sticks are exactly the same. Trusting that one thing equals another can be very exposing.

The ideas of equals supports the theoretical notion of equality. Justice is based on equality. If nothing is equal, what do we mean by the term justice? The related term, 'equity' implies inconsistency in treatment to ensure an equitable outcome. For example the handicap found in golf is to even the playing field and give everyone an equal chance at success by giving the more advanced players an initial disadvantage. How can we offer 'an equal footing' to different peoples? The concepts within mathematics need further expansion.

There are some internal paradoxes within mathematics for example the representation of some ideas $1/3 + 2/3 = 0.\underline{3333} + 0.\underline{6666} = 0.\underline{9999}$ (even given the understanding that these decimal fractions continue for infinity) this suggests that 1 = 0.9999… Is one equal to almost one?

Examples

To take a few simple examples of how a new understanding of the concepts in mathematics might assist a move toward sustainability.

Example 1:

If we simply look at number we miss the fundamental point. To consider the nuclear threat. What is the real difference between 8500 for Russia and 300 for France, when we are talking about a nuclear arsenal? The world has experienced two atomic warheads, each able to destroy cities, the current available fire power is 700 times that of the total fire power of World War II. What is the reality of the difference between hundreds or thousands of cities destroyed? We can all be destroyed through the deployment of nuclear war heads. Discussing the number of warheads is meaningless. Even discussing who has what, is also not relevant. While any nuclear war head exists there is a danger to us all. The use of mathematical models as the major tool of reconciling danger, assessing relative risk, and determining the future is problematic in that it undermines the human understanding that any one nuclear bomb is bad enough (the formula or computer model won't accept that 1 = 8000).

The absolute belief in the formulas generated by humans that approximate reality may potentially lead to the destruction of humanity. The world is a set of structured couplings (Maturana, 2002) and feedback mechanisms that are enormously delicate and complex and part of an open system and therefore exposed to many random variables.

What is needed is an inclusive and heterogeneous approach to problem solving that celebrates diversity and seeks input from a wide range of sources, celebrating the richness of group wisdom and futures thinking (Weinberger, 2011).

Example 2:

How can we establish a sustainable world without considering these proportions, for example: a cup of coffee takes 11365 gallons of water per 1 pound or 2500 litres per 100grammes of coffee (Orange County Water District). It is not sufficient to consider the numbers and simply worry about the water shortage. While this is important, the real concern is for individuals who do not have sufficient water, to drink, wash or cook with. This is a real, immediate and personal situation, made clear by consideration of the daily existence of real people. Hiding behind the numbers there lies a real human story. Doubling, halving or manipulating these numbers does not tell the real impact on life without water. Statistics and numbers are often used to create a picture, and we rely on their use to keep us informed. The way that numbers are presented as representing reality, can undermine the real experience and situation.

We need to incorporate other wisdoms to our decision making processes, to that any one narrow paradigm does not dominate and produce further replication of the problems solving seen to date. Fixing one problem while creating another has been characteristic of the 19th and early 20th centuries. Time now to develop other 'soft skills' and centres of knowledge that do not strive to be 'right' so much as to be 'righted'.

Example 3:

A J curve is a simple mathematical tool to show exponential growth. While this model is easy enough to draw on paper, it cannot exist in the finite world. In the end all things come to an end. The concept of infinite growth is applied in two major world challenges: economics and population increase.

A growth model is espoused by the world's finance leaders as important to ensure employability, and productivity within the industrial world. Simply the model cannot exist. In the real world there is a downward pressure of growth due to a limited capacity. There is a finite endpoint. There is irreversible destruction that occurs during the growth, and a point past which growth (in terms of pollution, over mining, depletion of stock, extinction and so on) cannot be reversed.

The population curve is also a J-curve. The statistics below may be considered in isolation as a set of numbers, or may have genuine impact on the human, and Earth story that they hide.

- Half of all the world's life human births are alive today
- At 9:24pm on 24/11/2014 there were 7,276,488,058
- By 2024 there will be 8 Billion people
- The population of the Earth has doubled in my lifetime. (approx. 1970 = 3 Billion, 1999 = 6 billion, 2011 = 7 billion)
- If it is your birthday and you are 60 years old today (Happy Birthday); on the day you were born 2,690,969,572 people were alive - almost 3 times as many people

> are alive now and creating rubbish, and consuming the world's minerals at a much worse rate

How can a J curve have no limit, in a world bound finitely? We all recognise emotionally what a crowded world might be like, but the numbers on a graph do not recognise experiences.

The curve needs to be demonstrated as a theoretical model only, not taken as a descriptive nor prescriptive pattern.

The mathematics of the future may need to involve fuzzy logic (Klir & Yuan, 1995) at an earlier time in the development of ideas; indicating that statistical uncertainty may be a part of the way that we learn mathematics rather than as a black and white absolute.

Conclusion

A new approach to mathematics might be helpful. Encouraging an ecological approach to mathematical thinking, and at the same time encouraging children to challenge established beliefs rather than accepting the dogma. The ecological approach demands an integrated treatment of curriculum, so that mathematics is taught alongside and integrated with human experience.

An ecological approach moves beyond systems (M.J. Wheatley, 2010; Margaret J. Wheatley & Kellner-Rogers, 1998) developing notions of inter-dependence, networks and relationships (Bateson, 1972) expanding the realm of complex forces that co-exist alongside human endeavours.

There remains some ambiguity among commentators (Gallopín, 2006) as to the specific relationship of economics and socio-ecologies. "Persistent disagreement both as to the interpretation to be given to sustainability, and as to the relation between ecological and economic sustainability, has hindered the development of an ecological economics of sustainable resource use." (Common & Perrings, 1992)

Currently mathematics embraces un-sustainable ideas of perfection, predictable patterns, infinite dimensions, perfect relationships, and absolute truths. Mathematicians espouse and expose the danger of 'knowable' and 'correct' bodies of knowledge. In reality some of these bodies of knowledge are not static, they are potentially 'unknowable' and they may at best represent only partial truths.

Our minds tend to rationality (but never quite make it). Rationality – represents a natural limit to the mind; close enough for most of the time, but just like the jumping into the door exercise, the rational mind, cannot ever understand the irrational world. The natural world remains stubbornly irrational. Is it potentially dangerous, or at least misleading to map our sense-making brain onto a world that was not 'designed' for us to understand?

Mathematics is a wonderful tool, an organising principle of logic, patterns, rules and relationships. It does not generate our world. We might need to keep developing ourselves and challenge even more deeply held beliefs to move forward in our collective understanding of how we can support an ailing world; starting with the pressing need for ecological sustainability. The alternatives to mathematics recognise that humans are part of complex adaptive systems. We need complex adaptive technologies and processes to mimic ecology, and protect the complex nature of society and our natural world.

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Logic and Inequalities: A Remedial Course Bridging Secondary School and Undergraduate Mathematics

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This paper is an informal discussion of a remedial lecture course that I plan to teach in the 2015–16 academic year in the Foundation Studies programme in my university. The course is set at the level bridging GCSE Mathematics as it is taught to students up to the age of 16 in secondary schools in England, and undergraduate mathematics courses for ages 18+. It does not overlap A Level Mathematics and Further Mathematics as they are taught in England for ages 16 to 19.

Introduction

This paper is an informal discussion of a lecture course that, subject to approval, I plan to teach next academic year in the Foundation Studies programme in my university. The paper expands its earlier version [1]. The proposed course is an update of the course that I taught every autumn since 2003.

In England, Foundation Studies courses in mathematics are intermediate Year Zero courses offered to students who were conditionally accepted to study at the university (mostly in Engineering and other STEM degree programmes), but who had not studied, or had not got good grades in, A Level Mathematics, or to students from overseas whose secondary school diplomas are not recognised in England. My particular course is large: 350 students, about 80 of them are from overseas.

My course serves as an introduction to the language of mathematics (well, to its English dialect), and to mathematical thinking.

Outline of the course

The course comprises 22 lectures of 50 minutes each, twice a week over 11 weeks, plus one tutorial class a week.

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Sets (6 lectures). Sets and their elements, equality of sets. The empty set and its uniqueness. Finite and infinite sets. $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$. Subsets: union, intersection and complement. Venn diagrams. De Morgan's Laws. Boolean algebra of sets.

Logic (8 lectures).
Propositional Logic. Statements and connectives. Conjunction, disjunction, negation, conditional, their interpretation in human languages. Truth tables. Material implication. Contrapositive and converse to conditional statements. Logical equivalence and tautologies. Boolean Algebra of Propositional Logic.
Predicate Logic. Predicates and relations. Universal and existential quantifiers. Some basic logic equivalences of Predicate Logic.

Proof. Proof of statements of the form $(\forall x)p(x)$ and $(\exists x)p(x)$ Proof of conditional statements

$$(\forall x)(p(x) \rightarrow q(x)).$$

Contrapositive and converse. Proof by contradiction. Proof by induction and computation by recursion.

Inequalities (8 lectures). Inequalities. Solution of inequalities containing unknown variables. Linear inequalities with one or two variables, systems of linear inequalities with two variables. Graphic representation of the solution sets of inequalities. Some simple problems of linear optimisation in two variables. Quadratic inequalities with one variable. Methods of proof for inequalities.

Backgrounds and justification

Why Logic? A very pragmatic justification

The new course contains almost everything necessary for mathematically competent writing and handling macroses for time-dependent EXCEL spreadsheets—and not much else. Time-dependent spreadsheets are the daily bread of practical computing in engineering and business; they are the principal mathematical tool of project management.

A reasonable grasp of propositional logic is useful for learning programming languages and for design and use of digital circuits—skills which are necessary for almost all engineering disciplines.

Inequalities are crucially important for applications of mathematics

To give just a few examples,

- Anything which contains the word "estimate"' in its name, whether it is in Engineering or in Economics, is based on inequalities.
- Anything which contains the word "approximation" in its name, whether it is in Engineering or in Economics, is based on inequalities.
- Anything which contains the word "optimisation" in its name, whether it is in Engineering or in Economics, is based on inequalities.

In addition,

- Inequalities are mathematical tools for control of errors in measurement and in experimental data, as well as for handling rounding errors in computations.
- Statistics is all about inequalities.
- Perhaps most importantly, the instinctive feel of inequalities makes the basis of quick "back-of-the-envelope" estimates and "guesstimates," the essential part of engineering thinking.

In words of Bertrand Russell,

> *Although this may seem a paradox, all exact science is dominated by the idea of approximation. When a man tells you that he knows the exact truth about anything, you are safe in inferring that he is an inexact man. Every careful measurement in science is always given with the probable error … every observer admits that he is likely wrong, and* ***knows about how much wrong he is likely to be***. [Emphasis is mine. – AB]

Inequalities are badly taught at school

Markov's Inequality:

If X is any nonnegative random variable and $a > 0$ then $\mathbf{P}(X \geq a) \leq \mathbf{E}(X) / a$,

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the first but fundamental result of the theory of random variables—and the basis of the entire Statistics—is no more than a primary school level observation about inequalities and can be formulated as an arithmetic "word problem" about anglers and fish:

> 50 anglers caught on average 4 fish each. Prove that the number of anglers who caught 20 or more fish each is at most 10.

A solution is simple. Assume that there were more than 10 anglers who caught at least 20 fish each; then these 10 anglers caught together more than $20 \times 10 = 200$ fish—a contradiction.

Unfortunately, we cannot expect that all students entering English universities are able to produce this argument. There are two reasons for that:

- This is a proof from contradiction—and this is why basic proofs from contradiction are part of the course.
- The argument requires simultaneous handling of two types of inequalities, "x is more than y," denoted $x > y$, and "x is at least y," denoted $x \geq y$.

Alas, I many times met students who were asking me questions of that kind:

> *How can we claim that* $3 \geq 2$ if *we already know that* $3 > 2$?

This fallacy is a symptom of a dangerous condition—logical deficiency. Handling inequalities demands stronger logical skills than mechanical manipulation of equations.

Moreover, inequalities are frequently more important than equations! For example, besides the equation for a straight line in the plane,

$$ax + by - c = 0,$$

closely related inequalities

$$ax + by < c, \quad ax + by \leq c, \quad ax + by \geq c, \quad ax + by > c$$

are no less important: they describe the way the line cuts the plane in two halves (and therefore have natural applications, say, in computer graphics). To give just one example, here is a simple problem:

> Answer without sketching graphs: do points (1,3) and (-2, 4) lie to the same side off the line $2x + 3y - 1 = 0$ or belong to the opposite sides?

Mathematical logic allows us to see connections between inequalities and equations which play an important role in many practical problems.

The natural affinity of the theory of inequalities and elementary logic

Inequalities fit happily into the course which starts with sets and logic not only because they need logic, but also because, in a way of reciprocity, systems of simultaneous inequalities provide accessible material for learning and applying techniques of logic, deduction and proof. Basic Boolean Logic: conjunction, disjunction, negation, comes into play very naturally. A system of two simultaneous inequalities is the *conjunction* of inequalities, the solution set of the system is *the intersection* of the solution sets of individual inequalities.

The inequality $x^2 > 1$ is *equivalent to the disjunction* of inequalities $x > 1$ and $x < -1$

$$(\forall x)(x^2 > 1 \leftrightarrow (x > 1 \wedge x < -1)),$$

and the solution set of $x^2 > 1$ is *the union* of the solution sets of $x > 1$ and $x < -1$:

$$\{ x \mid x^2 > 1 \} = \{ x \mid x > 1 \} \cup \{ x \mid x < -1 \}$$

The *negation* of the inequality $x^2 > 1$ is $x^2 \leq 1$, and the equation $x^2 = 1$ is the *conjunction* of inequalities $x^2 \geq 1$ and $x^2 \leq 1$:

$$(\forall x)(x^2 = 1 \leftrightarrow (x^2 \geq 1 \wedge x^2 \leq 1)).$$

Even more remarkable (and the reason why quadratic inequalities need to be discussed not only for their practical importance, but also as an illustrative material for logic), that the inequality $x > 1$ *implies* the inequality $x^2 > 1$,

$$(\forall x)(x > 1 \rightarrow x^2 > 1),$$

but $x^2 > 1$ *does not imply* $x > 1$,

$$\neg(\forall x)(x^2 > 1 \rightarrow \mathrm{x} > 1),$$

or, rewriting this statement in a logically equivalent way,

$$(\exists x)((x^2 > 1) \wedge \neg(x > 1));$$

in plain language, it means

there exists x such that $x^2 > 1$ but $x \leq 1$.

Systems of simultaneous inequalities are *predicates* – *unary*, in case of systems of inequalities in one variable, and *binary* – if we have two variables.

Is Logic too hard?

At an elementary restricted level – no, it is not. Logical formulae that I gave as examples might appear to be excessively complex. But the logical connectives $\neg$ (negation), $\wedge$(conjunction), and $\vee$(discjunction) are routine operators in computer coding. The *universal quantifier* $\forall$ and the *existential quantifier* $\exists$, if used sparingly, help students to develop sharper reasoning skills. In an example above, the statement

it is not true that $x^2 > 1$ implies $x > 1$

translates into symbolic notation as

$$\neg(\forall x)(x^2 > 1 \rightarrow x > 1),$$

and is logically equivalent to

$$(\exists x)((x^2 > 1) \wedge \neg(x > 1)),$$

which, in plain language, means

there is x such that $x^2 > 1$ but $x \leq 1$.

Without prior exposition to basic symbolic logic, many students would have difficulties in understanding that the statement

(A) it is not true that $x^2 > 1$ implies $x > 1$

is the same as

(B) there is x such that $x^2 > 1$ but $x \leq 1$.

My aim, of course, is to help my students to eventually see that (A) and (B) are the same without resorting to logical symbolism. Graphic representation of inequalities (and therefore some basic set-theoretic thinking) is a useful stepping-stone: both (A) and (B) are equivalent to saying that

the set of solutions of $x^2 > 1$ is not a subset of the set of solutions of $x > 1$.

I will restrict the use of alternating quantifiers to gently introduced single change cases, $\forall\exists$ and $\exists\forall$. Indeed, the $\forall\exists$ combination already has to be handled with great care, it triggers the explosion of infinity:

for *every* number *there is* a bigger number,

$$(\forall x)(\exists y)(x < y).$$

I will definitely avoid the notorious $\forall\exists\forall$, the perilous stumbling block of the ε-δ language of the real analysis.

Some even more general methodological observations

A bit of cognitive science and didactic considerations

An additional link between inequalities and logic is provided in example that I am systematically using in the logic part of the course: (pre-)order relations and their logical combinations appearing as kinship relations in human societies. If $F(x,y)$ denotes "x is the father of y", the difference between

$$(\forall y)(\exists y)F(\mathrm{x,y})$$

and

$$(\exists x)(\forall y)F(x,y)$$

becomes instantly self-evident. Kinship is remarkably self-evident, and for very deep cognitive and evolutionary reasons—already apes and even monkeys have sophisticated kinship systems.

The wonderful book *Baboon Metaphysics* [2] provides some astonishing evidence—and please notice that the book contains a formal definition of strict linear order:

> *The number of adult males in a baboon group at any given time ranges widely, from as few as 3 to as many as 12. Regardless of their number,*

> *however, the males invariably form a linear, transitive dominance hierarchy based on the outcome of aggressive interaction (a linear, transitive hierarchy is one in which individuals A, B, C and D can be arranged in linear order with no reversal that violate the rule 'if A dominates B and B dominates C, then A dominates C'). Although the male dominance hierarchy is linear, transitive, and unambiguous over short periods of time, rank changes occur often (Kitchen et al. 2003b), and a male's tenure in the alpha position seldom lasts for more than a year.* [p. 51]

> *Like males, female baboons form linear, transitive dominance hierarchies. There, however, the similarity ends. Whereas male dominance ranks are acquired through aggressive challenges and change often, female ranks are inherited from their mothers and remain stable for years at time. Furthermore, most female dominance interactions are very subtle. Although threats and fights do occur, they are far less common and violent than fights among males. Instead, most female dominance interactions take the form of supplants: one female simply approaches another and the latter cedes her sitting position, grooming partner, or food. The direction of supplants and aggression—and the resulting female dominance hierarchy—is highly predictable and invariant. The alpha female supplants all others, the second-ranking supplants all but the alpha, and so on down the line to the 24th- or 25th-ranking female, who supplants no one.* [p. 65]

It is truly remarkable to what degree the concept of linear order is self-evident to humans. But anyone who taught freshmen knows that the concept of equivalence relation is considerably harder. The reason is that transitivity of dominance is obvious at the level of the monkey bits of our brains. The transitivity of equality is a much later, in evolutionary terms, social construct. In the powerful scene in the film *Lincoln* [3], Abraham Lincoln says to his astonished aids:

> *Euclid's first common notion is this: Things which are equal to the same thing are equal to each other. That's a rule of mathematical reasoning. It's true because it works. Has done and always will do. In his book, Euclid says this is 'self-evident.' You see, there it is, even in that 2,000-year-old book of mechanical law. It is a self-evident truth that things which are equal to the same thing are equal to each other.*

The scene is a fiction, but it is very true in spirit to a number of well-documented quotes from Lincoln where he uses references to Euclid as a logical and rhetoric device (see, for example, a real quote from him later in this paper).

I refer to Abraham Lincoln as to authority to prove that the level of abstraction in the course is set carefully and balanced by self-evident nature of principal examples.

Cultural and linguistic issues

Students in the course come from a variety of socioeconomic, cultural, educational and linguistic backgrounds. Just at a level of basic notation, I have to deal with students who, in their school mathematics, were using two different symbols for multiplication:

$$2 \cdot 3 = 6 \quad \text{and} \quad 2 \times 3 = 6,$$

and three different symbols for division:

$$6/3 = 2; \quad 6 : 3 = 2; \quad 6 \div 3 = 2.$$

Some countries use decimal point:

$$\pi = 3.1415\ldots,$$

while others prefer decimal comma:

$$\pi = 3{,}1415\ldots,$$

this list can be easily continued.

Even more obstructive are invisible differences in the logical structure of my students' mother tongues. For example, the connective “or” is strictly exclusive in Chinese: “one or another but not both”, while in English and Russian “or” is mostly inclusive: “one or another or perhaps both”, with an occasional slip into the exclusive mode: “her *or* me!” Meanwhile, in mathematics “or” is always inclusive and corresponds to the expression “and/or” of the bureaucratic slang. But this is only the beginning: in Croatian, for example, there are two connectives “and”: one *parallel*, to link verbs for actions executed simultaneously, and another *consecutive* (and some signs of that can be detected in Russian).

Articles are a special issue; many languages have no definite articles, and therefore, in the very first lecture, some time is devoted to the detailed discussion of the expression

the empty set.

The list can be continued almost indefinitely; the key point here is that teaching elementary logic to a ethnically and linguistically diverse audience inevitably becomes an exercise in multiculturalism, and a lecturer has to be prepared to face the challenge.

Material implication

The principle of *material implication* states:

it is true that

- "false" implies "true", or, in symbolic notation,

$$F \to T,$$

and

- "false" implies "false", which becomes, in symbolic notation,

$$F \to F.$$

Material implication is the hardest part of cultural accommodation of my students. This principle is unacceptable and even morally offensive to many of them, and for deeply rooted cultural reasons; their objections have to be treated with great respect.

Indeed, the principle of material implication follows from the expectation that the statement

$$P \to (Q \to P)$$

is always true, regardless of the validity of P or Q. The latter has meaning

If P is true, it is true regardless.

There is a Russian proverbial saying about card games:

Ace is an Ace, even in Africa,

and a more general Hungarian one:

A gentleman is a gentleman, even in Hell.

The glorious tautology

$$P \to (Q \to P)$$

is the proclamation of independence of thinking:

> if something is true, it is true regardless of what others say about it;

and, after specifying the meaning of the word "others", becomes anti-authoritarian and subversive:

> if something is true it is true regardless of what the (traditional and absolute) authority says about it.

Teaching logic means teaching thinking, and, as soon as you start teaching thinking, you get into the heat of dead serious ideological debates. A lecturer has to tread this path very, very cautiously, taking great care with every his/her step.

Psychological support and the ``radical remediation''

By the nature of the course, some of my students had unhappy experiences with school mathematics. As a lecturer, I have to give them moral and psychological support. It helps that the content of the course deviates from the standard school curriculum—this allows me tell my students, that they are given a chance to start from clean page and leave their fears and frustrations behind. Using the words coined by Roman Kossak, it is a "radical remediation" of mathematical learning.

Finally, a bit of metamathematics

As the great mathematician and philosopher Alfred Tarsky had explained in his seminal work [4] on elementary geometry,

> Logic $\cup$ (quadratic and linear inequalities over the reals) $\subseteq$ *Euclid's Elements*.

The course described in this paper, is, from the formal metamathematical point of view, the Euclidean Geometry in (unrecognisable for a lay person) disguise smuggled back into the secondary school level curriculum.

Therefore the material in the course could be made as rich as the Euclidean Geometry, with all the opportunities for rigorous definitions and rigorous (and not very complicated) proofs. The course can be slightly extended and reinforced to arm the student with the same power of deduction as was demonstrated by Abraham Lincoln (and this time these are his own words):

One would start with confidence that he could convince any sane child that the simpler propositions of Euclid are true; but, nevertheless, he would fail, utterly, with one who should deny the definitions and axioms. The principles of Jefferson are the definitions and axioms of free society. And yet they are denied, and evaded, with no small show of success. One dashingly calls them 'glittering generalities'; another bluntly calls them 'self-evident lies'; and still others insidiously argue that they apply only 'to superior races'. [5]

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Disclaimer

The author writes in his personal capacity and the views expressed do not necessarily represent position of his employer or any other person, organisation or institution.

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[5] A. Lincoln, *Collected Works*, 3:375, quoted at C. S. Morrissey, Spielberg's Lincoln: Politics as Mathematics, *The Catholic World Report*, December 19, 2012, http://www.catholicworldreport.com/Item/1822/spielbergs_ilincolni_politics_as_mathematics.aspx; Morrissey, in his turn, quoted G. Havers, *Lincoln and the Politics of Christian Love*. Columbia, Missouri: University of Missouri Press, 2009, p. 72.

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Solving Application Problems Using Mathematical Modelling Diagrams

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This paper describes three simple practical problems solved by three university lecturers to demonstrate applications of mathematics or illustrate some issues in mathematics education. The problems were of different levels from primary to secondary school although the settings were at a university. All lecturers either explicitly or implicitly used the four step diagram for solving the problems: real problem => mathematical model => mathematical solution => real solution => real problem. While modelling the lecturers formulated inadequate mathematical models without the constraints of the variables involved. That led to contrasting the 'mathematical solution' and the 'real solution' which might have resulted in incorrect perception of the role of mathematics in real life among some students. The author suggests that the contrast could be avoided by using the three step diagram instead of the four step diagram for the process of mathematical modelling of such application problems: real problem => mathematical model => solution => real problem.

There are many diagrams that illustrate the mathematical modelling process of solving real life problems. Some of the diagrams are very detailed and complicated. An example of such a diagram is represented in Figure 1:

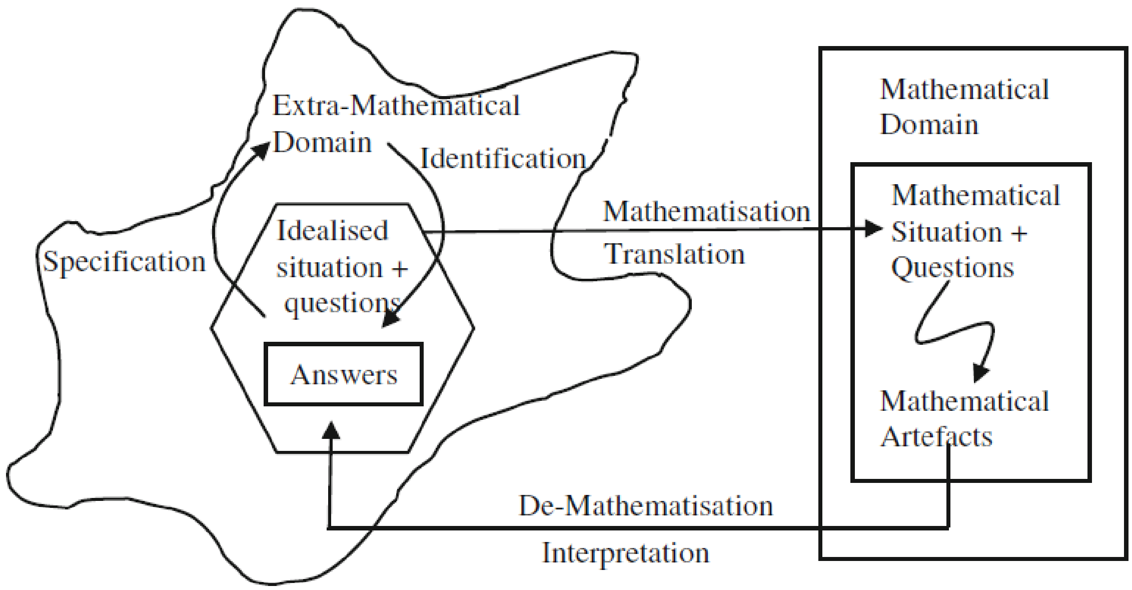

Figure 1. Mathematical modelling process (from Niss, 2010, p.44).

Many textbooks on calculus use much simpler versions like the one represented in Figure 2:

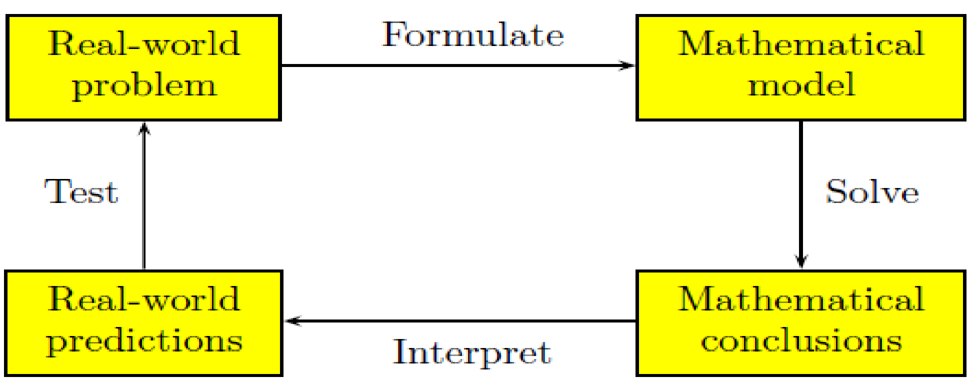

Figure 2. Mathematical modelling process (from Stewart, 2010, p.25).

Even a simpler version is very common in textbooks and in teaching practice:

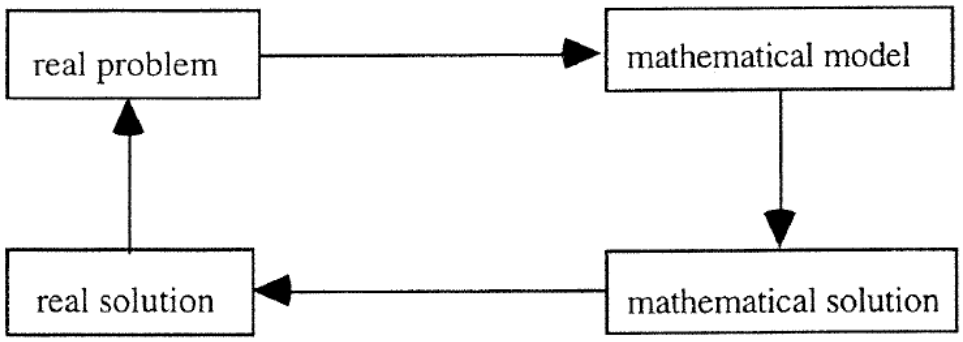

Figure 3. Four step mathematical modelling diagram.

The above diagrams differentiate the mathematical solution and real solution. Indeed, when we solve a real life problem we make assumptions, construct the mathematical model and solve it to obtain the solution of the mathematical model. After that we need to test that solution on the original real life problem and if it doesn't fit we need to adjust assumptions and/or refine the model and repeat the cycle again until we are satisfied with the outcome.

This paper is based on the author's observations of teaching mathematics and mathematics education courses while on sabbatical at a partner university. Within a month he came across three situations when the lecturers used the four step diagram represented in Figure 3 for solving simple application problems.

Problem 1. How many buses does it take to transport 145 people if each bus can take only 20 people?
The lecturer's commented that the 'mathematical solution' was 7.25 but it didn't suit us and the 'real solution' was 8.

Problem 2. There are three sorts of alloy bars. Their weights and contents of iron and copper are: A (10, 2, 1), B (15, 1, 3), C (15, 4, 3). How many of each of the alloy bars do we need to meet the following constraints: total weight 25; total amount or iron 9 and total amount of cooper 4.

After solving the system of the linear equations

$10x + 15y + 15z = 25$
$2x + y + 4z = 9$
$x + 3y + 3z = 4$

as a mathematical model and obtaining the answer (1, -1, 2) the lecturer commented that the 'mathematical answer' was not realistic and the 'real solution' of the problem did not exist.

Problem 3. A stone was thrown at an acute angle from the top of a 60 m high cliff. In 2 seconds the stone was in its highest point of 80 m. Assuming a quadratic relationship between the height and time, find the time when the stone lands on the ground.

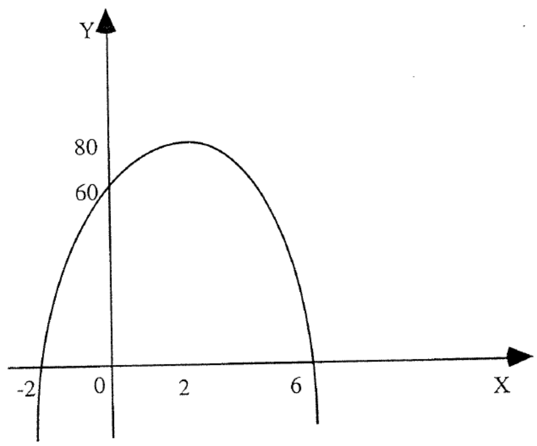

Figure 4. Sketch of the trajectory of the stone for problem 3.

Here the mathematical model was: find abscissas of the intersection points of the parabola with the *x*-axis. That led to two answers: -2 and 6. After the discussion, the 'real solution' (6) was chosen from the 'mathematical solutions' (-2 and 6).

It is clear that all three lecturers contrasted the 'mathematical solution' to the 'real solution' in the above problems. The range of the variables was taken into consideration only at the stage of adjusting the 'mathematical solution' to the 'real solution'. These contrasts could be easily avoided if the constraints on variables were included into the mathematical models. For example, the mathematical models of the three problems might be:

Mathematical model of problem 1. Round up the number 145/20.
Mathematical model of problem 2. Find non-negative solutions of the system of the linear equations.
Mathematical model of problem 3. Find a positive *x*-intercept.

In such simple application problems there is no need to make assumptions, test and refine the models if they are constructed adequately – by taking into account the range of the variables involved. In most cases what is needed is just an interpretation of the solution by adding the units (kg, cm, etc.). In some cases care should be taken of reasonable accuracy of the solution (often students write the answer obtained from a calculator accurate to 5-7 decimal places).

For such simple application problems like problems 1-3 the following 3 step diagram might be an alternative:

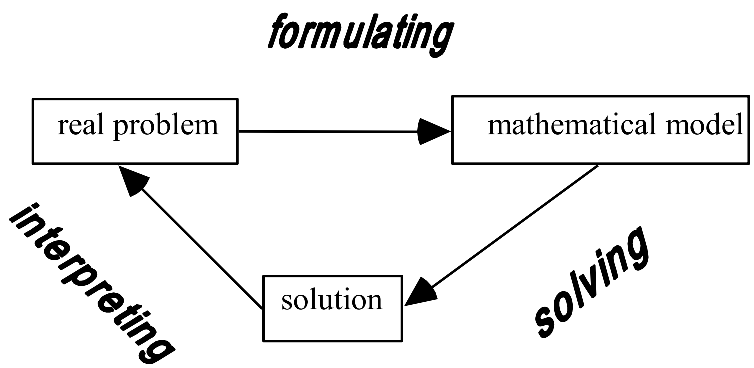

Figure 5. Three step mathematical modelling diagram.

The advantage of this diagram compared to the four step diagram is that it might prevent the psychological discomfort among some students caused by contrasting the 'mathematical solution' and the 'real solution'. The matter of adequacy of a mathematical model is very important even in simple application problems especially for non-maths students. Otherwise we might face a situation that happened on a chemistry lesson when a student 'proved' that 1 plus 1 is not always 2. He mixed 1 litre of water with 1 litre of spirit and found that the total volume of liquid is not 2 litres! (It is a well-known fact that the mixture of water and spirit has more compact molecular structure).

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Using Common Sense in a Mathematical Modelling Task

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Abstract

The paper compares responses of students (novices) and lecturers (experts) to questions regarding differences in predictions from 3 different mathematical models of a real-life problem. The problem was based on the data of the spread of SARS (Severe Acute Respiratory Syndrome) in Hong Kong in 2003. The models were based on the same data but they gave very different predictions of the spread of the disease. Although the majority of the students used common sense compared to the lecturers who used their knowledge and experience in explaining the differences, the proportions of correct answers were not far apart. It might suggest that the use of common sense in modelling real-life problems can be a good starting point in dealing with some modelling issues.

Introduction

We believe that even simple mathematical modelling activities can be beneficial for students. We agree with Kadjievich who pointed out that "although through solving such … [simple modelling] … tasks students will not realize the examined nature of modelling, it is certain that mathematical knowledge will become alive for them and that they will begin to perceive mathematics as a human enterprise, which improves our lives" (Kadijevich, 1999).

In many cases a major purpose for mathematical modelling of a phenomenon is to make predictions. Taking into account uncertainty, variety of possible models and a number of assumptions in each model the task of prediction cannot have the "correct" answer. This fact alone can confuse many students. This paper investigates students' opinions regarding differences in predictions from 3 different models based on the same real data. The task given to the students might look very simple. They neither needed to build a model nor to solve the given models. All they needed to do was to read the given real life problem, look at the predictions from 3 different models and give their reasons for the differences in the predictions. We tested one of the modelling competences described by Kaiser in (2007): "Relating back to the real situation and interpreting the solution in a real-world context". We also gave the same task to university lecturers who teach mathematics or mathematical modelling courses. Our idea was to compare the responses of the students and lecturers. The main research question was to investigate possible patterns within each group and also similarities and differences between the two groups when they do the same modelling task. In particular, to which extent the two groups use their intuition, common sense and past experience explaining the differences in predictions from 3 familiar models.

The theoretical framework of this study was based on the works of Haines and Crouch (2001, 2004). A measure of attainment for stages of modelling has been developed in (Haines & Crouch, 2001) The authors expanded their study in (Crouch & Haines, 2004) where they compared undergraduates (novices) and engineering research students (experts). They suggested a three level classification of the developmental processes which the learner passes in moving from novice behaviour to that of an expert. One of the conclusions of that research was that "students are weak in linking mathematical world and the real world, thus supporting a view that students need much stronger experiences in building real world mathematical world connections" (Crouch & Haines, 2004). This was consistent with the findings from the study by Klymchuk & Zverkova (2001) on possible practical, not cognitive reasons for students' difficulties linking mathematics and real world. Referring to that study Crouch and Haines wrote: "…students across nine countries all tended to feel that they found moving from the real world to the mathematical world difficult because they lacked such practice in application tasks" (Crouch and Haines, 2004).

The Study

Three easy models of the real epidemic of SARS in Hong Kong in 2003 - linear, exponential and logistic - were offered as a student project in calculus in (Hughes-Hallett, et al., 2005). Although the models were based on the same data, they gave very different predictions of the spread of the disease. We asked two groups of people, students and lecturers, to explain the differences in predictions from the three models in an unfamiliar (for students) context. The students' group consisted of first-year undergraduate students majoring in engineering from a German university and second and third-year students majoring in applied mathematics from a New Zealand university. Ninety questionnaires were distributed and 48 responses were received so the response rate was 53%. It was a self-selected sample. We systematized and grouped students' answers into different categories according to the nature of their responses. We used either the key words or exact quotes to name the categories. Some students gave multiple responses to some of the questions and some students did not answer all the questions. The lecturers' group consisted of university lecturers from different countries who teach mathematics or mathematical modelling courses. Some of them were involved in research on teaching mathematical modelling and applications. Some of the lecturers were from the same universities as the students participated in the study. Thirty eight questionnaires were distributed and 23 responses were received so the response rate was 63%. It was a self-selected sample. We systematized and categorized the lecturers' answers in the same way as the students' answers.

The questionnaire given to the participants of the study is below.

The Questionnaire

Please read the case below and answer the questions. You don't need to solve anything.

In 2003 a highly infectious disease SARS spread rapidly around the world. Predicting the course of the disease – how many people would be infected, how long it would last – was important to officials trying to minimise the impact of the disease. A number of mathematical models of the spread of SARS were developed to make the predictions. Below are three simple models of the spread of SARS in Hong Kong. We measure time t, in days since March 17, the date the World Health Organization (WHO) started to publish daily SARS reports. Let $P(t)$ be the total number of cases reported in Hong Kong by day t. On March 17, Hong Kong reported 95 cases. We compare predictions for June 12, the last day a new case reported in Hong Kong (87 days since March 17). The constants in the differential equations were determined using WHO data from 17 to 31 March (15 days).

A Linear Model $\frac{dP}{dt} = 30.2$, $P(0) = 95$. The prediction for June 12 was 2722 cases.

An Exponential Model $\frac{dP}{dt} = 0.12P$, $P(0) = 95$. The prediction for June 12 was 3,249,000 cases.

A Logistic Model $\frac{dP}{dt} = P(0.19 - 0.0002P)$, $P(0) = 95$. The prediction for June 12 was 950 cases.

The actual number of cases on June 12 was 1755.

Please answer the following questions:

1. What were possible reasons for the differences in the predictions from the three models above?
2. On what were your reasons from question 1 based (e.g. your experience in modelling, common sense, etc.)?
3. What could make the predictions more accurate?

Students' Responses

The students' categorized responses are presented below.

1. What were possible reasons for the differences in the predictions from the three models above?

Different models (16), lack of biological factors (10), different ideas of the speed of spread (8), isolation of infected people (8), population density (6), different assumptions of cases per day, report of cases is not correct (3), different infection rates (3), counter actions, for example pharmaceuticals, different side conditions (1), different assumptions for each model (1), probability of onset (1), people developed immunity (1), the predictions are theories, which are different from the reality (1), not enough data (1).

2. On what were your reasons from question 1 based (e.g. your experience in modelling, common sense, etc.)?

Common sense (19), mathematical knowledge and experience in modelling (7), both modelling experience and common sense (3), the given information (1), idea of spread of disease (1), I have never seen such problems in mathematical context before, so I don't know exactly, how to solve it (1), reality, never a constant number of persons will be sick (1), my knowledge about curves of elementary functions (1).

3. What could make the predictions more accurate?

Use experiences from studies of other epidemics, in other regions (14), use more data (7), more knowledge of the virus (3), look for preventive steps, compulsory registration (2), improve data collection (1), average value of cases from 7 days (1), a constant showing the rate of infections (1), side effects like number of travellers to and from Hong Kong (1), information of medical doctors or scientists for the course of disease (1), a study of people behaviour and their health state (1), more facts (1), evaluation of the models (1), the logistic model looks more realistic and it could be improved by using more variables (1), set up a limit of resources (1), adjust the models results to the reality all the time (1), compare the first 2-3 days to find the initial condition (1).

Lecturers' Responses

The lecturers' categorized responses are presented below.

1. What were possible reasons for the differences in the predictions from the three models above?

The models (19), different ideas of the spread of the disease, certain factors were not considered (2), the models were developed for other epidemics, SARS does not fit (1), the assumptions are not the same in all three models (1), did not consider the spread style of the disease (1), infinite number of predictions exist (1).

2. On what were your reasons from question 1 based (e.g. your experience in modelling, common sense, etc.)?

Experience in modelling (13), common sense (5), both modelling experience and common sense (3).

3. What could make the predictions more accurate?

More data (6), a better model (3), better parameters estimation (3), knowledge about infection mechanism and other factors e.g. travelling routes, social patterns (2), more accurate analysis of influencing factors (2), a deeper understanding of how infectious disease spread (1), the parameters in all the models must be the same (1), distribute the observing time in intervals and use different models in different intervals (1), use learning methods (1).

Analysis of the Responses

After consultations with professional mathematicians specialising in epidemic modelling we estimated percentages of appropriate answers to questions 1 and 3 in both groups. The results are presented in the table below. 'CS' means 'common sense' and 'Exp' means 'experience'.

		Question 1	Question 2				Question 3	Question 4	
	N	Appropriate	CS	Exp	Both	Other	Appropriate	Yes	No
Students	48	73%	56%	20%	9%	15%	74%	9%	81%
Lecturers	23	92%	24%	62%	14%	0%	90%	64%	36%

Table 1. Summary of the findings from the questionnaire.

The majority of the students had no or very little experience in mathematical modelling. The closest activity to real mathematical modelling for them was solving application problems. To our surprise the students did well in both modelling questions 1 and 3. They were not much behind the lecturers giving 73% appropriate reasons for the differences in the predictions from the models versus 92% given by the lecturers. They were not much behind the lecturers giving 74% appropriate ways to improve the accuracy of the predictions in the models versus 90% given by the experts. This is consistent with the findings by Haines and Crouch (2001, 2004) where the authors found that sometimes novices exhibited aspects of expert behaviour although they were not consistent in doing so. In particular, in their study on self-assessment and tutor assessment they found that students were almost as good as tutors in assessing group (project) presentations on modelling and so they could recognize modelling behaviour in others. It is the consistency demonstrating expert behaviour that perhaps puts the lecturers ahead.

In question 2 the reverse polarity on the answers by the students and the lecturers was anticipated: the students relied more on common sense (56%) rather than on experience (20%) compared to the lecturers (24% on common sense and 62% on experience). Apart from lack of modelling experience by the students one of possible reasons for that reverse polarity might be elements of the lecturers' behaviour where they were reluctant to put their responses down to common sense, preferring to classify it as experience. After all they have invested a great deal of time in mathematics/modelling.

Based on the participants' comments in the questionnaire and follow-up interviews with some of them we attempted a comparison of the processes used by the students and the lecturers in terms of links between the mathematical world and the real world in a similar way it was done in (Crouch & Haines, 2004). We took the first "level a) where there was clear evidence that the participants took into account the relationship between the mathematical world and the real world" (Crouch & Haines, 2004). The students referred explicitly to that relationship in 65% of cases (though not always in a correct way) whereas the lecturers in 20% of cases. The lecturers tended to concentrate more on the mathematical aspects of the models probably implicitly assuming that relationship. One of the possible reasons might be that the lecturers used their experience in modelling and knowledge in mathematics much more than their common sense whereas the students relied more on their common sense and life experiences lacking the experience in mathematical modelling.

Conclusions

This study indicates that in spite of lack of experience in real mathematical modelling, students can effectively use their common sense and general knowledge of mathematics to evaluate some modelling issues dealing with prediction. The responses at a more general level indicated that both students and lecturers would have preferred to include more parameters in the model to make the modelling more realistic and intuitive, i.e., to have a theoretical basis for the modelling that included hypothetical rates of spread, infection mechanisms, etc.

We are very aware of the limitations of the study. It was intended as a pilot study to check our assumptions and share the findings with the mathematics education community. Future work should explore students' and lecturers' (or novices and experts according to Haines and Crouch, 2004) responses to more sophisticated mathematical models that allow for the adjustment of parameters to optimize the output from the model.

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CREATIVITY AND BISOCIATION

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Abstract. *Creativity research has its roots in reflections upon eminent mathematics creating original and useful mathematical products. Its extension into the mathematics classroom leaves open many definitions and understandings of its value and role in education of gifted students. The extension of creativity to ordinary students who do not see themselves as gifted or the demoncratization of creativity is explored through the lens of Koeslter's theory of bisociation and the relationship between bisociation and reflective abstraction the mechanism of learning proposed by Piaget is put forth.*

Key words: bisociation, reflective abstraction, incubation, illumination

HISTORICAL DEVELOPMENT OF CREATIVE RESEARCH

The historical development of creativity is typically traced to the original and creative work of eminent mathematicians. Inherent in this view is measurability through the end result or the product definition (Liljedahl,2011, 2013) in which the community of mathematicians is to verify the truth of as well as to judge the usefulness and originality of the idea presented. The extension of creativity to the mathematics classroom is often traced to the work of Jaques Hadamard who synthesized the stages the Gestalt psychology Wallas to explain the setting in which the "Aha" moment occurs through a process-stage that begins with incubation in which the solver tries diligently but unsuccessfully to obtain the goal to the instantaneous insight of the illumination stage. Hadamard it would appear believed that the distinction between the creative genius and a creative individual learning mathematics that is known to the community but not him or herself is only a matter of degree. Yet research on incubation and the transition to illumination is scarce in mathematics education and the mechanisms poorly understood (Sriramen et al, 2011) Perhaps one reason is the view that concepts of incubation and illumination are considered, " …arachaic Gestalt constructs…" (Juter & Sriramen, 2011) which as Piaget (1977) and Koestler (1964) would note has fallen from fashion due in part to its overly strict focus on not only an instantaneous but also complete insight into problems and structures that in cognitive development takes places in repeated flashes of insight or eureka moments that build on one another and occur as a result of continued effort by the individual learner. Another related reason is that the affective component of creativity is too difficult a concept for educational researchers and psychologists to measure objectively, indeed the intuitive nature of the creative experience is often tied to a self transcendent force in which inspires a spontaneous "Aha" or Eureka moment s simultaneous with a complete solution to the problem. One can only imagine the comments a pedagogy based upon complete solutions mystically popping into students head would receive.

Transition from genius to gifted students

The transition from the so-called genius understanding of creativity, to a more inclusive definition and understanding that allows for research into creativity in the mathematics classroom especially in regards to research in problem solving (Silver, 1997) has spurred a significant amount of research and a multitude of definitions and what creativity in discovering what is known to others means Mann (2006).

One might also point out that definitions and understandings used to measure creativity with gifted students within the mathematical classroom can be viewed as being developed to distinguish between gifted and non-gifted or ordinary students. Leikin (2009b) notes two formulations of creativity in mathematics educational research that have been used to assess an individual's propensity for creativity in the mathematics classroom or student giftedness. The first is the ability for convergent and divergent thinking due to J. Guilford. "Convergent thinking involves aiming for a single correct solution to a problem, whereas divergent thinking involves the creative generation of multiple answers to a problem or phenomena, and is described more frequently as flexible thinking." Her review also notes the definition suggested E. Torrance i.e. the capacity of an individual for flexibility, fluency, novelty and elaboration. "Fluency refers to the continuity of ideas, flow of associations, and use of basic and universal knowledge. Flexibility is associated with changing ideas, approaching a problem in various ways, and producing a variety of solutions. Novelty is characterized by a unique way of thinking and unique products of a mental or artistic activity. Elaboration refers to the ability to describe, illuminate, and generalize ideas." (p.129) Using this document

Transition from gifted to ordinary students

Yet these characteristics such as fluency with mathematical content, the ability to view and understand various different approaches to a problem solving specifically including those not modelled by the instructor can be used to identify and characterize students not in science, mathematics, engineering or related fields who are enrolled in mandatory first year math course (Blair et al. 2006)

The lack of educational research on creativity with the general population has been pointed out by Srimanen et al (2011) “The role of creativity within mathematics education with students who do not consider themselves gifted is essentially non-existent” (p.120). Clearly if such research is to be done what we (Prabhu and Czarnocha, 2014)) shall refer to as the ‘democratization’ of creativity one needs a definition in which to measure creativity that does not exclude ordinary students. Before reviewing Koestler theoretical framework for creativity and its relationship to the issues raised it is useful to consider on why one might be interested in creativity with students who do not consider themselves gifted in mathematics. Liljedahl (2013) studies students he classified as ‘resistant’, “Many of the students would describe themselves as math phobic, math-incapable or a combination of the two. They usually have negative beliefs about their abilities to do mathematics, poor attitudes about the subject, and dread the thought of having to take a mathematics course”(p.256) exposing them to creative pedagogy and asking them to write about their experiences. He reported that they experienced illumination or “Aha” moments both while listening to the instructor and while working problems out for themselves the latter being much more common and that such experiences frequently did alter their attitudes towards mathematics. One of the basic insights of Prof. V. Prabhu is that students who struggle with mathematics especially in underserved populations frequently have a negative affect towards mathematics and perhaps the only way to reach such students is through a pedagogy that guides and exposes students to their own creative potential, a social contract (Surrazy & Noventná, 2013) that encourages them to own their learning process what she would refer to as a handshake between the instructor and the student. Furthermore, the lack of such pedagogy propagates a society in which teachers such as those described by Liljedahl (2013) (before his research project) will pass on their own negative view of mathematics to children who will follow their lead and then pass on such negative views to friends and children, in such a way an entire society itself begins to accept that is cannot do mathematics and that such a situation is a natural state.

KOESTLER THEORETICAL FRAMEWORK FOR CREATIVITY

Koestler (1964) describes the mechanism of creativity in terms of an (hidden) analogy between two or more previously unrelated frames of reference: “I have coined the term *bisociation* in order to make a distinction between the routine skills of thinking on a *single plane* as it were, and the creative act, which … always operates on *more than one plane*” (Koestler, 1964, p. 36).

The terms matrix and code are defined broadly and used by Koestler with a great amount of flexibility. He writes, "I use the term *matrix* to denote any ability, habit, or skill, any pattern of ordered behaviour governed by a *code* or fixed rules" (p. 38).

> The *matrix* is the pattern before you, representing the ensemble of permissible moves. The *code* which governs the *matrix*…is the fixed invariable factor in a skill or habit; the *matrix* its variable aspect. The two words do not refer to different entities; they refer to different aspects of the same activity. (Koestler, 1964, p. 40)

Thus, for Koestler (1964), *bisociation* represents a "spontaneous flash of insight ... which connects previously unconnected matrices of experience" (p. 45). That is the"…transfer of the train of thought from one matrix to another governed by a different logic or code …" (p. 95)

The focus of the Act of Creation Theory is on the bisociative leap of insight, that is, an *Aha!* moment, or a *moment of understanding*, **—** a phenomenon that contains both an affective component of the 'Aha' moment and a cognitive component of the synthesis of two previous unrelated matrices of thoughts the hidden analogy as Koestler would refer to it. These components in so far as they can be observed amongst the general population suggests Koestler's framework as suitable for measuring and analyzing the creative aspect of self discovery during learning.

REFLECTIE ABSTRACTION AND BISOCATION

In order to observe bisociation within the self discovery of mathematics during the many leaps of insight some obtained when listening to the instructor or during working as a class, in a small group or individually it is useful to reflect upon the relationship between Bisociation and Koestler's framework for creativity with reflective abstraction the mechanism used by Piaget to explain learning. Like the mechanisms of bisociation it is based upon conscious reflection and abstraction of solution activity.. According to Piaget, construction of schema and cognitive change:

> …proceeds from the subject's actions and operations, according to two processes that are necessarily associated: (1) a projection onto a higher level (for example, of representations) of what is derived from the lower level (for example, an action system), and (2) reflection, which reconstructs and reorganizes, within a larger system, what is transferred by projection. (Piaget & Garcia, 1989, p. 2)

Dubinsky (1991) employs Piaget's mechanisms of reflective abstraction on actions and processes within the process/object duality of concept development. Dubinsky considers *interiorization*, *coordination*, *generalization*, *reversibility* and *encapsulation*. Interiorization involves the internalization of processes or actions, while reversibility involves reflection upon the inverse of a known process. Coordination of two processes is an integral part of reflective abstraction as defined above when after projection the learner reflects upon and begins to reconstruct and reorganize his original knowledge coordinating it with processes and structures available in the larger structure. The original projection of knowledge onto the higher plane and the reflection required during the coordination of this projected knowledge with processes of the higher plane generalizes and abstracts the learners knowledge to the higher structural level. This work of Dubinsky has laid the foundation for APOS or action-process-object-schema theory in which the learner transitions between through three types of knowledge or understanding the action level: actions, processes, and objects, which are themselves organized into schema. "An action is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli which the individual perceives as external. " (Cottrill et al 1996,p.2) In contrast to a process in which the individual is in control and can reflect upon the transformation. Processes are the result of the Interiorization of actions and at the process level an individual can employ the reflective abstraction processes of coordination and reversibility. The object level is obtained through the final encapsulation processes.

Simon et al. (2004) ascertain that such reflective abstraction, or more specifically, coordination, is brought about by comparisons of the solution activity and its effect in relation to the expected outcome:

> Thus, within each subset of the records of experience (positive or negative results), the learners mental comparison of the records allows for recognition of patterns, that is, abstraction of the relationship between activity and effect. Because both the activity and the effect are embodiments of available conceptions, the abstracted activity-effect relationship involves a coordination of conceptions … Note the activity and the effect are conception-based mental activities, our interpretation of Piaget's notion of coordination of actions (Simon et al., 2004, pp. 319-320)

This interpretation of Piaget's mechanism of coordination broadens the scope of reflection upon actions to reflection upon solution activity. In Koestler's framework for problem-solving, the experiential mental records are matrices, and coordination occurs initially between the individual's record of possibly related matrices and the given problem situation. A straightforward analogy, or association, is characterized by an isomorphic mapping, or a projection, of a matrix to the current situation, resulting in assimilation. When no such equivalent matrix is found, the solver searches for a matrix that, although not previously considered completely analogous, is suddenly seen as giving structure and meaning to the present situation. The hidden analogy comes to light. When this realization is led by intuition, Koestler refers to it as bisociation, and likens it to the process often exercised by mathematicians and scientists to exploit the structures of one discipline to give meaning, through analogy, to a problem in another.

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