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Contextualized Examples in Constructivist Mathematics Pedagogy

Ayalur Krishnan and Max Tran
Department of Mathematics and Computer Science,
Kingsborough Community College, CUNY, USA

Abstract
We argue that one can adopt a constructivist approach to teaching basic mathematics by using contextualized examples arising in everyday life. Mathematical ideas are hidden in many everyday situations and using these situations to make the ideas more tangible aids in their grasp and assimilation. We argue a correlation between linguistic and mathematical ability and suggest that using contextualized examples can aid in implementation of the Writing Across the Curriculum (WAC) program within CUNY.

1. Introduction

Constructivist pedagogical practices have been of interest as a scholarly subject for the past century and a half. The epistemological premise of constructivist pedagogy is that knowledge and meaning is accrued through interaction with the world through individual experiences and ideas. We see this play out in children who learn concepts heuristically and attempt to test their learned ideas in the context of their experiences. The Montessori approach to elementary education, for instance, prioritizes this natural learning over direct instruction. This approach presumes the existence of various innate psychological "human tendencies" such as order, exploration, self-perfection, abstraction etc. and attempts to provide a framework where there is efficient actualization of these innate characteristics [1].

It is obviously of interest to see how constructivist pedagogies influence the learning of the social sciences, since the subject matter in these disciplines has a significantly more subjective flair, than, say, the physical sciences. It is reasonable to imagine that the circumstances of one's childhood and one's idiosyncratic bent of mind will have a significant influence in how concepts in the subjects of, for example, history, economics, sociology or philosophy are received and internalized. This aspect then also affords an opportunity to investigate to what extent, if any, the basic concepts of these subjects are independent of individual perceptions. We are thankfully in the position of investigating the constructivist approach in what is perhaps the most objective of all the scholarly disciplines, mathematics. Without going into deeper philosophical considerations about the ontology of mathematical objects, it is safe to say that insofar as elementary understanding of numbers, proportions, shapes, relations etc. is concerned, ontological differences in individual perceptions are not crucial where basic conceptual and algorithmic aspects are concerned. In mathematics, then, one can efficaciously apply the constructivist approach for a grasping of the basic concepts. Here we show how contextualized examples using concepts arising in everyday life, and using everyday language, one can achieve efficient and comprehensive conceptual understanding of basic mathematical ideas.

In his article "Reconstructing Mathematics Pedagogy from a Constructivist Perspective" [6], Martin Simon investigates the instructor's role in facilitating the trajectory of students' mathematics thinking and learning. In it he refers to the French mathematics pedagogue Guy Brousseau's ideas on the theory of "didactic situations". Simon says: "Brousseau asserts that part of the role of the teacher is to take the non-contextualized mathematical ideas that are to be taught and embed them in a context for student investigation. Such a context should be personally meaningful to the students, allowing them to solve problems in that context, the solution of which might be a specific instantiation of the idea to be learned." We heartily approve of this sentiment. From experience in the classroom, we often hear the refrain "What's the purpose of all of this?" If mathematical ideas can be contextualized appropriately and their relevance to everyday life highlighted, students are inclined to expend interest and energy in learning the subject. As we argue in this article, contextualized problems arising in everyday life provide an appropriate framework to instill a sense of the utility of mathematics.

2. Language as Primary Source and the WAC Program at CUNY

Mathematics is curiously situated both within and without natural language. One needs natural language to speak about mathematics, but one obviously does not need mathematics to investigate natural language. Every mathematical symbol used has a corresponding word in natural language that is used to refer to it. In this sense, one recalls the oft-quoted sentiment that mathematics itself is a language. In fact, it's a supremely efficient one: imagine what the size of a calculus textbook would be if one were to go through the entire text and replace every symbol with the corresponding word in natural language! This aspect of the interplay between mathematics and natural language finds its earliest expression in elementary mathematical concepts such as numbers, fractions, proportions etc. In these early situations, the natural language is used to talk about and fathom the physical world, and in so doing, mathematical ideas begin to seamlessly percolate into the psyche. Thus, in learning the words for natural numbers, one realizes that the natural numbers continue forever, without end. By the time one has arrived at calculus or group theory, of course, one has internalized the rules of formation and expression in the pure mathematical language, and one does "pure mathematics". Indeed, many concepts of higher mathematics are pure abstractions, devoid of counterparts in the physical world. But in the beginning stages of mathematical learning, one does not have the facility to consider mathematics as a purely formal language. It is in this area that constructivist pedagogy can have the greatest impact in facilitating conceptual understanding, to be followed eventually by algorithmic fluency.

Our students come from very diverse backgrounds. Often times we instructors cannot take for granted even a consistent vocabulary set to be used for instruction, given the vast differences in cultural backgrounds of our students. This is a well-known issue in education. The famous oarsman-regatta question in the SAT comes to mind [2]. Many years ago an analogy question in the SAT asked students to find the pair of terms which was most similar in their relationship as the pair of words runner-marathon. The correct answer, oarsman-regatta, would typically be beyond the experiential sphere of those who did not grow up in a wealthy environment where rowing would be a popular sport. Such language issues are of greater relevance in the humanities. In mathematics, however, we do not need to presuppose awareness of specific cultural paradigms in order for a grasping of the concepts. Everyday examples devoid of cultural specificity can be advantageously used in aiding mathematics instruction. We give below a variety of examples to show how this can be done.

The issue is not so much linguistic comprehension, but that of comprehension in a particular language. Thus, it is conceivable, and even expected, that students who grew up in non-English speaking cultures would have trouble with English idioms, for example. But it is reasonable to assume that all students understand the concepts relating to everyday situations arising all over the world without regard to country or culture. It is these kind of examples we focus on, and it is in this sense that we consider language as a "primary source". In other words, "language" does not refer to any particular language, but a certain linguistic facility, which could be in any language that one uses to think in and comprehend the world. Mathematical concepts are hidden in most everyday situations. The concepts do not have to be always numerical or geometrical: logic plays a role in many situations.

The Writing Across the Curriculum (WAC) program in CUNY is motivated by the considerations above. Graduating from CUNY with a college degree now requires a student to have taken and passed at least one WAC course. These are courses in which there is a substantial writing component. Instructors teaching the WAC courses have to be certified by going through a training process where they learn how to incorporate writing assignments into their courses. In the current instance, the first author is WAC certified, and routinely teaches WAC courses in mathematics. The writing assignments get the students to engage with mathematical symbols and ideas and help them realize that once the symbols are rendered in ordinary English, their meaning is not as arcane as the symbolism appears. This engagement with the natural language that is going on "behind the scenes" in mathematics helps the students to get a better appreciation for the use of, and need for, mathematical symbolism. The examples appearing here can all be converted to writing assignments for a WAC course.

In the context of linguistic vs. mathematical ability, some readers might be interested in a recent research report that suggests that the correlation between reading and mathematics ability at age twelve has a substantial genetic component [3].

3. Contextualized Examples

We assume that the basic concepts up to addition and subtraction and the corresponding algorithms are understood. It is perhaps impossible to articulate in natural language how we grasp these basic concepts. The idea of number is perhaps “hard-wired” into our brains. Philosophers generally take cognizance of the natural numbers as a starting point for meta-mathematical investigations. Philosophers such as Kant hold that numbers are *a priori* concepts. In this context, even as adults we find it problematic to explain our blind faith in the elementary algorithm for the addition of two numbers. Thus, we promptly agree to the statement that 36 + 72 = 108, but no one would seriously claim to have made a mental construction of grasping 36 (different) objects, then mentally juxtaposing them with another 72 (different) objects, and mentally verifying that this new aggregate now actually has 108 objects. On the contrary, we use highly sophisticated concepts such as induction on a meta-mathematical level to convince ourselves that the algorithm for addition always yields a correct result if the basic steps are carried out without error. The basic steps, corresponding to the rules for adding single digit numbers, can easily be verified using actual physical objects.

Beginning mathematics students typically start to have difficulties when we move to multiplication, division, proportions, fractions etc. since typically the approach to teaching these subjects is to stress the algorithm, rather than give a sense of the origin and utility of these concepts in everyday life. We outline here a variety of examples which treat elementary concepts and ground them in everyday experience. This approach can be carried out in regards to most elementary mathematical concepts, with a little thought. Once one gets to more advanced mathematical concepts, this approach is unnecessary (though still useful) since by then our brains have “magically” learned to appreciate the form and function of pure mathematics. We attempt to address the beginning student, such as perhaps someone in the remedial mathematics courses at CUNY, and we hope the constructivist approach will make that student cultivate a friendly disposition to the subject.

We start with a simple and elegant problem.

> Two pipes A and B supply water to a swimming pool. Pipe A by itself can fill the pool in 4 hours, and pipe B by itself in 6 hours. How long will it take for the pool to be filled if both pipes are kept open?

The beginning student faces a variety of difficulties with this problem. It is not immediate how this problem can be approached. The first step is to grasp the problem from a real-world perspective. If the real-world implication of the problem is grasped properly, then one would realize that the final answer, whatever it maybe, has to be less than 4 hours. For someone who has not grasped the problem at all, “both pipes” suggest addition, with a resulting answer of 10 hours. The constructivist approach in class would be to first start a discussion on how a solution can be attempted. As ideas are suggested and their error or correctness is discussed, the problem becomes more familiar and grounded in experience. Visualization of a pool with two pipes gushing water can be suggested if one wants to incorporate a "contemplative practices" aspect to the instruction. At some point, one idea will either be discovered by the class to be key, or else revealed by the instructor as key: *what part of the pool is filled by pipe A by itself in one hour*? The answer, $1/4^{th}$, will already cause difficulty to some students. Once this is carefully grasped, the solution is in sight. Pipe B by itself fills $1/6^{th}$ of the pool in one hour, and therefore together $(1/4 + 1/6) = 5/12^{th}$ of the pool is filled in one hour when both pipes are open. From here, the idea the whole pool is filled in 12/5 hours, or 2 hours and 24 minutes, is yet another idea that is not straightforward, but can be linked to the earlier idea of how to calculate what part of the pool was individually filled by each of the pipes, and shown to be a reverse process. It can be emphasized that this is a problem that can actually arise in real life if one is fortunate enough to live in a house with a pool, and that it requires mathematics for its solution.

In the foregoing problem, fractions played an important role. The idea of the Least Common Denominator can similarly be shown to have a real life significance as in the following problem:

> You have one candy bar. How will you give someone a total of a half plus a third of the candy bar?

This problem can first be motivated by simpler examples, say involving a half and a third of an hour. Thus, if one task takes half an hour to complete, and another task takes a third of an hour, it is clear that the two tasks together take fifty minutes to complete. When it comes to a candy bar, a half is obtained by dividing the bar into two equal parts and choosing one part, and a third is obtained by dividing the bar into three equal parts and choosing one part. So far so good, but if there is only one bar, how could one simultaneously divide it into two and three equal parts? The idea that a half of the bar can equally be obtained by dividing the bar into *six* equal parts and taking *three* of them, and similarly for a third of the bar, at once shows the power of the Least Common Denominator. Just by one appropriate division of the bar into a certain number, one can now simultaneously obtain a half and a third of the bar. Of course, the denominator does not have to be "least": any common denominator will do, and this leads to further exploration and verification of the traditional algorithms for adding fractions, reducing fractions etc.

Multiplication and division show up naturally in percentage problems, and are an excellent way to solidify these concepts. To start with, the etymology of the word "percent" shows that we are considering quantities *per hundred*. A regular shape such as a square or a circle gives a good visual analogy. Thus, to calculate thirty percent of a pizza pie, one would divide the pie into a hundred equal parts, and then take thirty of those parts. To calculate say 40% of 25, one is essentially dividing 25 into 100 parts, and then taking 40 of those parts. That is, one is doing the computation (25/100)*40. But typically the computation is introduced as (0.40)*25, which is really the same thing, but conceptually less illuminating. If 30% of a quantity is 60, this means that 30 parts of that quantity make up 60, and hence that 1%, or 1 part, is 2, and hence the full quantity which is 100%, or 100 parts, is 200. The interplay between multiplication and division is clear in these simple examples. But the following problem tests the patience of many students:

> After a 20% discount, the price of a shirt is $40. What is the original price of the shirt?

The student who is inclined to jump to computations before having grasped the problem will doubtless calculate 20% of 40 to get 8, and add this answer to 40 to get the answer of $48, a wrong answer. In this problem, visualization is key. One has to visualize that there is a starting quantity and that 20% of that starting quantity is removed, so that 80% of that quantity remains. If this is grasped, then as previously, 80% equals 40, whereby 1% equals 0.5 and hence the original quantity, which is 100%, equals 50, giving the correct answer of $50. This problem also shows the pitfall of formulas. While formulas are very handy, understanding and correctly substituting for the variables in the formula is key. One common formula taught in doing percentage problems states: is/of = %/100. A careless application of the formula, using numbers provided in the problem also gives a wrong answer. The constructivist approach favors an intuitive solution to problems by grounding them in everyday experience rather than attempting to apply formulas right off the bat. Once the problem is understood in the proper perspective, one discerns a natural path between what is stated and what is asked for.

The following problem treats proportions.

> Nine people stranded in a desert have enough water to last them for eighteen days. Three of them are abducted by aliens in UFOs. How many days will the water last the remaining six people? Assume that every person consumes water at the same rate.

For students familiar with setting up proportion problems as the appropriate fractions separated by the equals-to sign and cross-multiplying to find the answer, the process will yield a wrong answer unless they observe that the problem is not one of direct proportion, but one of inverse proportion. Engagement with the problem in a real-world scenario (except for the aliens in the UFO part) suggests that the water should last longer, since now there are lesser number of people. But this problem can also be attempted without setting up equations. The content of the last sentence of the problem can be used to introduce a variable denoting the rate of consumption per person per day. If this variable is called “w”, then the total amount of water present, prior to abduction, is 9*w*18. If now there are only six people remaining, then they consume 6*w amount of water per day, and hence the water would last them (9*w*18)/(6*w) = 27 days. As in the previous problem, engagement with the problem can suggest intuitive answers without having to invoke potentially tricky formulas.

Mathematics also deals with logical principles. Some of these principles are often used implicitly in solving problems, without one’s being aware of their usage. Some principles are treated as axioms, in that the principle is seen to be evidently true when the concepts involved in the formulation of the principle are studied. The Pigeonhole Principle is one such. Thus, in a class consisting of say forty students, it has to be the case that some two of the students will share the same month of birth. Discussion as to why this has to be true then leads to the question of how many students would have to be present so that some two of them necessarily share the same day and month of birth. This illustrates a distinctly *mathematical* or *analytical* way of thinking.

To illustrate with another example, when discussing exponents, one can introduce the idea of ancestors and how the number of ancestors for each person grows exponentially when considering both parents of each ancestor. After calculating some initial exponents, it is quickly seen that these numbers start to grow very quickly. A question for discussion at this point would be to investigate if these numbers can keep on growing, yielding more and more ancestors. Interesting discussions ensue, ultimately resulting in the realization that this unbounded growth cannot continue, leading to a simple "proof" that all humans must have evolved from a set of common ancestors. On the same topic, another problem is that a (very lucky) person finds one penny on day one, two pennies on day two, four pennies on day three, eight pennies on day four, sixteen pennies on day five, thirty-two pennies on day six, and so on; how long would it take for this person hit $1 Million? Before doing any actual calculations, if the class is asked for their estimate of the time, it usually is a large number, since $1 Million is a large amount. But the actual answer (28 days) is somewhat shocking, and illustrates how quickly exponents grow. A rather incomprehensible and fascinating fact in this context is this: a Googol is 10 raised to the power of 100, and a Googolplex is 10 raised to the power of a Googol; while a Googol can obviously be written down, a Googolplex cannot even be written down: there is not enough space in the universe even to write down this many number of zeroes! [4].

The grounding of examples in reality does not have to be restricted to elementary mathematics. Concepts arising in middle and high school mathematics are also shown to arise naturally in real life. Thus, interest calculations, mortgages and annuities, and population growth are naturally modeled by arithmetic and geometric series. Set operations such as power sets, subsets and the like can be exemplified using distribution of a basket of candy amongst recipients. The harmonic series provides a good example of how our intuition can lead us astray when it comes to very small quantities. A classic problem in pre-calculus states:

> Suppose you start from the foot of a mountain at 12 noon and climb to the top of the mountain following a certain path. You camp at the top overnight and start back from the top of the mountain the next day at 12 noon, and follow the same path down. Show that there will be some particular time at which you would have been at the same point on the path on both your journey up and on your journey down.

This is not intuitively clear immediately, but once the graphs of distance vs. time are drawn for the journeys up and down, and shown to intersect, the intuition becomes clearer as to why the stated conclusion must be true. After all, intuition is also not a fixed, but grows more sophisticated with more learning. A similar problem in calculus states that on any great circle around the world, for any scalar quantity which varies continuously such as temperature, pressure, elevation, or carbon-di-oxide concentration, there will be two antipodal points that share the same value for the variable [5]. Intuitively this is far from obvious. As a final example, a simple problem such as determining the shape of a fence, given a fixed amount of rope, so as to maximize the area included *requires* calculus for its solution.

4. Conclusion

There is no silver bullet when it comes to constructivist techniques for mathematics instruction. Bauersfeld [7] states: "The fundamentally constructive nature of human cognition and the processual emergence of themes, regularities, and norms for mathematizing across social interaction, to bring the [psychological] and the social together, make it impossible to end up with a simple prescriptive summary for teaching. There is no way towards an operationalization of the social constructivist perspective without destroying the perspective." That said, instructors develop an intuition for what works and what doesn't, from experience in the classroom. In our experience, grounding problems in everyday life has the best effect when it comes to retaining the attention and interest of students.

There is a subtle interplay in mathematics between form and content, and, curiously, this is mirrored in the issues surrounding the "math wars" [8]. It is quite impossible to ground the calculation of, say 45.67*9.876, in "everyday experience". Here one must learn the multiplication algorithm and practice it many times to do similar computations without error. At the same time, simpler examples of the same kind can be grounded in everyday life. Thus, form and content feed off of each other: algorithmic fluency and conceptual understanding develop hand-in-hand. In the beginning stages, however, the conceptual understanding is somewhat harder to instill. This is where contextualized examples grounded in everyday life can be of the greatest efficacy.

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