Constructivized Calculus: A Subset of Constructive Mathematics

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Abstract
The purpose of this paper is to demonstrate the importance of Constructive Mathematics in today’s college mathematics curriculum. In the spirit of the philosophies of LEJ Brouwer and Errett Bishop, a history of constructive mathematics will be presented. Constructive mathematics gives numerical meaning, and quantifies abstract concepts. The main goal of this paper is to identify how constructive calculus, which is based on constructive mathematics, can serve as a tool for engineers, scientists, computer scientists, economists, business majors, and applied mathematicians. Classical or traditional calculus contains many “existence theorems” which states that a quantity exists but these theorems do not indicate how to find this quantity. The constructive version of the ‘existence theorems’ describes how to find the quantity and as a result how it can be used for practical purposes.

Introduction to Constructive Mathematics
Constructive mathematics finds its roots in the intuitionist philosophy of Leopold Kronecker and L.E.J. Brouwer. Starting in 1907, Brouwer strongly criticized classical mathematics about its idealism and lacking in numerical meaning. Fifty years later Errett Bishop resurrected the intuitionist philosophy by Brouwer but referred to as the constructivist movement. According to Bishop (1970)

“…It appears then that there are certain mathematical statements that are merely evocative, that make assertions without empirical validity. There are also mathematical statements of immediate empirical validity, which say that certain performable operations will produce certain observable results…Mathematics is a mixture of the real and ideal, sometime one, sometimes the other, often so presented that it is hard to tell which is which…”

Constructive mathematics seeks reliable results of activities that lead to computational manipulations. Therefore the constructivist’s role included eliminating the idealism which has come to define the very existence of the traditional mathematics. In order to do this, many definitions and concepts must be reformulated starting with the existing classical mathematical definitions and concepts. This process is diametrically opposed to starting from void and creating or developing an entirely new branch of mathematics. For example, constructivized calculus relies on the classical calculus. To present a formal system under the intuitionist/constructivist philosophy, a series of finite steps are needed.
to derive a numerical result. Although it is necessary to use a finitary process, it is not sufficient. Proofs of theorems must also be presented constructively.

**Comparison Between Classical and Constructive Mathematics**

Comparisons between classical and constructivized mathematics have focused on the ideal versus the real, with idea of quantifying in constructive mathematics as opposed to merely accepting the existence in classical mathematics. Bishop (1968) gives an elegant difference

“…Constructive existence is much more restrictive than the ideal existence of classical mathematics. The only way to show that an object exists is to give a finite routine for finding it, whereas in classical mathematics other methods can be used…Theorem after theorem of classical mathematics depends in an essential way on the limited principle of omniscience, and therefore not constructively valid…”

An example of an existence theorem in integral calculus is the Mean Value Theorem for Integrals which states:

*If* \( f \) *is a continuous on the closed interval* \([a, b]\) *then there exists a number* \( c \) *in the closed interval* \([a, b]\) *such that* \( \int_{a}^{b} f(x) \, dx = f(c)(b-a) \).

The theorem lacks the steps it takes to find the \( c \). Instructors and teachers of calculus must teach students the geometric interpretation of the theorem. In addition, it is necessary to teach students how to find the value of \( c \) using the definite integral of \( f(x) \); this requires using the functional value to find the independent variable now represented by \( c \).

The constructive version of the Mean Value Theorem for Integrals states:

*Let* \( f(x) \) *be continuous on* \([a, b]\) *then for* \( \forall \, \varepsilon > 0 \), *one can find* \( a < x < b \) *subject to* \( |f(x)| < \varepsilon \) *and

\[
\left| \int_{a}^{b} f(t) \, dt - f(x)(b-a) \right| < \varepsilon \text{ or } \left| f(x)(b-a) - \int_{a}^{b} f(t) \, dt \right| \leq \varepsilon
\]

On the contrary, the constructive version of the Mean Value Theorem for Integrals requires constructing a rectangle that has the same length as the interval \([a, b]\) but a height that depends on finding a \( c_i \)'s which represents the midpoints of intervals determined by the interval halving process such that the height is equal to \( f(c_i) \). Other examples of existence theorems in classical calculus are the Intermediate Value Theorem for Derivatives and Rolle’s Theorem.

**Importance of a Constructivized Calculus Curriculum**

The traditional calculus curriculum in the liberal arts programs emphasizes the two basic operations of differentiation and integration using algebraic and transcendental functions. The calculus definitions, concepts, theorems and principles required to solve these problems include sequences, series, countably infinite sets, continuous functions, least upper bound, convergence, Rolle’s theorem, the law of the mean, the intermediate value theorem and the fundamental theorems of calculus, which are idealistic and abstract in nature, lacking a direct process to describe the numerical value. The proofs of the theorems often require a step by step procedure to verify the hypothesis of the conditional
statement but whether there is always a correspondence to the set of real numbers or even the set of integers, has been determined to not be the case. The Constructive proofs must have a finite number of steps whereas the classical proofs do have this restriction. In addition, the Law of Excluded Middle and proof by contradiction cannot be used. These are common techniques in classical calculus. This weakness of the traditional calculus curriculum could be strengthened by implementing a constructivist calculus curriculum.

Even non-mathematics majors enrolled in calculus courses are expected to apply these concepts in various fields. For example, one of the problems of calculus determines the activity of a body in motion whereas other problems involve finding areas, volumes, arc lengths, center of masses, instantaneous speeds and optics. When a problem involves finding the largest/smallest (maximum/minimum) of a function in the form of a mathematical model in the fields social and natural sciences, calculus techniques are used. Students are more concerned with how to find quantities and less concerned with the theoretical basis of the techniques.

The goal of the college mathematics is to provide students with the mathematical skills needed for their chosen fields. These skills include but are not limited to developing quantitative skills, providing conditions to enhance the ability to think clearly, using technology and improving problem solving skills. The general education goals are to foster the development of students’ communication skills, learning skills and motivate them to move to higher levels of independent learning. These goals are based on the recommendations from the Mathematical Association of America (MAA), the National Council of Teachers of Mathematics (NCTM) and Society for Industrial and Applied Mathematics (SIAM).

Several majors require students to take calculus courses as part of their programs of study. In this role, the Mathematics Department provides this service. According to MAA, the specific skills that other departments need from the calculus sequence are to serve the needs of these disciplines through the use of computational techniques. These majors also need the mathematics to help support recent developments in their fields and enhance the use of technology in order to make the appropriate decisions. The MAA report also states,

‘…The increased technologically sophistication of the world we live in has created a need for a more mathematically and technologically literate citizenry. Mathematics courses, textbooks, and curricula changed dramatically, during the twentieth century often in response to the physics and engineering of the time. Further changes are needed in the twenty-first century to respond to the needs of the expanding set of disciplines for which mathematics now plays an increasingly important role…’

The report also describes other ways that Mathematics Departments have and can support programs in other disciplines.

‘…At some institutions, mathematics courses to support the programs of partner disciplines are taught within those disciplines… When this is the case, mathematical sciences departments can approach these departments to signal a willingness to be involved in the further development of the course. Such an overture may be especially
welcome when the partner departments are having difficulty staffing the courses. Often the reason the courses were originally developed because mathematicians were not interested or were not teaching the topics in a way or at a level that was appropriate for the students in the partner discipline…”

In most calculus classes, students who need to take the course are majoring in engineering, business, computer science and mathematics (STEM). According to the U.S. National Center for Education Statistics, of the students who received degrees in STEM majors except mathematics (≈ 22%) far outnumber STEM students who received degrees in mathematics (≈ 0.11%). The traditional calculus courses are more appropriate for goals of the mathematics major and only marginally meet the needs of the remaining STEM students. According to the ICMI (2008) study,

‘…It would be desirable that service mathematics courses enabled students to acquire a range of essential knowledge, skills, and modes of mathematical thought; each professional activity demands a particular mathematical literacy so mathematics courses must include applications, examples, modeling processes, etc. which motivate students…”

Some of the concepts, theorems, and their proofs in classical calculus lack numerical meaning and are non-computational. Therefore classical calculus is neither as effective nor as efficient a tool as it should be for the business, engineering science or computer science major. In computer science, constructive mathematics is extremely helpful for writing computer programs for use in the real world or to assist in research. It can also be used to determine the center of mass of objects. Constructive mathematics is also of interest to symbolic logicians. Bridger’s (2007) point clarifies this position,

‘…Not every student in Real Analysis is a math major, and in many schools, only a small percentage of math majors intend to do graduate work in mathematics. A modern course is populated by a wide range of students. Some are headed for careers in secondary education, while there is often a large contingent from the physical sciences and an even larger group from computer science. These students are in the course because they need or want more than a cookbook calculus course. Some need to know more about computability and calculability of floating-point numbers, hence more about the actual nature of the reals. They also need to know about continuity because they need to know about approximations; some need to know about convergence and improper integrals because they need to know about computing special functions and transforms…”

In practice, calculus instructors’ solutions to teaching calculus to so many non-mathematics majors is to adapt the curriculum by omitting certain topics or emphasizing procedures and methods while de-emphasizing theorems and their proofs. While this could work in some situations, the rigor of classical calculus suffers. To maintain the rigor of classical calculus while addressing the needs of non-mathematics majors, the constructivized version of calculus can replace it. Constructivized calculus is more general and rigorous than classical calculus. Errett Bishop, the pioneer of constructive analysis, was a liberal constructivist; his approach to constructivization was to only
substitute those statements in traditional mathematics which were non-constructive (i.e. existence theorems and proofs which utilized the Law of the Excluded Middle) with a constructive version. Alsina (2007) discusses the role of mathematics at the collegiate level,

‘…The central importance of mathematics in our technologically complex world is undeniable, and the possibilities of new applications are almost endless. But at the undergraduate level, little of this excitement is being conveyed to our students. Currently, attention is being focused on reforming calculus, the traditional gateway course into the undergraduate curriculum. No one is questioning the importance and beauty of continuous mathematics. However, reformed or not, calculus is one branch (and a highly technical one) of a very rich subject. We know the breadth and richness of our subject; how, then, do we expect the students who are starting their study to gain insights…’

Some students can gain an insight through constructive calculus because it will meet the needs of the technologically complex world for which we are preparing them.

The traditional calculus course can be revised to include the constructivized versions of the definitions of functions and limits, the concepts of continuity, the derivative and the definite integral, and Rolle’s, Intermediate Value, Mean Value Theorems and the Fundamental Theorem of Calculus. Students of calculus should be exposed to the constructive calculus curriculum. The students might face the challenges of understanding the differences between the two versions of calculus, particularly if they took previously took a calculus course. This is best summarized by Taschner (1998)

‘…Although severe differences between the traditional form of mathematics and the constructive access to analysis exist…It is merely to show the students how to switch back to the traditional beaten track…’

The traditional calculus student can use the same approach when shifting back and forth between concepts and theorems which are constructivized to other concepts and theorems which do not require constructivization.

Conclusion

Students majoring in engineering, economics, business, computer science and even physics can benefit greatly from a constructive calculus course. The concepts and theorems of a constructivized calculus would prepare the non-mathematics STEM students to apply these ideas in their respective areas of expertise. Another benefit of a constructive access to calculus is that students gain a feeling of what is really possible in mathematics – and what melts into the shadow of purely formal reasoning (Taschner, 1998). Most importantly, students also gain a deeper consideration of appropriate definitions. The following statement can summarize the significance of a constructivized calculus curriculum (Fletcher, 1998)

‘…It is an advantage of constructivism that it stimulates us to consider alternative mathematical treatments…constructivism permits alternative ways of handling infinity and infinitesimals…’
Several branches of mathematics have been researched and studied to be presented in the constructivist version such as algebra, topology, and analysis. Mathematics has always been a tool for other practical areas such as engineering, economics, business, computer science, biology, and physics, to name a few. The question of using constructive mathematics as a basis has been discussed recently in the works of several mathematicians such as Bridges (1999), Seidenberg (1974), Richman (1982), and Fletcher (2002). Because constructive mathematics emphasizes finding numerical meaning in a finite number of steps it would have a significant role in computers and technology, which defines the twenty-first century.

References
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