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UNIVERSAL AND EXISTENTIAL QUANTIFIERS

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Investigation into students learning of logical quantifiers is extended to include both universal and existential quantifiers. The paper extends the analysis of the data into root causes of student difficulties and with the help of innovative teaching-research interviews, which identify routes for the improvement of learning. This way, every research clinical interview of a student about mathematics can be enhanced, without loosing its scientific character, as it becomes a carefully crafted, research-based tutoring session, reducing time lag between research and practice to zero. Readiness for introduction the symbolic approach is analyzed. The work confirms the impact of natural language on the mathematical understanding of negation by identifying, during the interview, a source of misconception initiated from incorrect French/English translation.

Keywords: Keywords: Advanced Mathematical Thinking, reasoning and proof, post-secondary education.

TEACHING–RESEARCH NYC MODEL

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Introduction

This report continues the discussion of the development of mathematical reasoning, and in particular understanding and negating quantifiers "all" and "some" (Bardelle, 2011; Ferrari, P.L. 2004; Dubinsky, 1997). There are interesting reasons why the negation of quantifiers is full of challenges for students, to a larger extent than for example the negation of the conjunction and the disjunction or even the conditional. Transformation of the conditional to the equivalent disjunction reduces it to the negation of that disjunction. While (Bardelle, 2011) and (Ferrari, P.L. 2004) point to the impact of the natural language register upon understanding the meaning of mathematical statements, following the line of reasoning of, e.g. (Cornu, 1981, Mason and Pimm, 1984), Dubinsky and his co-workers initiated a large investigation program to understand learning of quantification assisted by the general Piaget-based theory of learning APOS (Action, Process, Object, Schema) with a moderate success (Dubinsky et al. 1988; Dubinsky, 1997; Dubinsky and Yiparaki, 1997). They point to student cognitive difficulties in constructing the meaning of the quantifiers and their negations. Meanwhile several empirical reports analyzed the difficulty of negating single quantifier statements (Lin et al, 2003, Zepp et al, 1987). The present work continues the path of empirical investigations motivated by the excellent (Bardelle, 2011) discussion, which analysed the difficulties of the general quantifier without including the existential one. We extend the analysis of difficulties to include the existential quantifier and have designed the set of questions (Appendix) similar to those used by (Bardelle, 2011). The report presents the results and analysis of the pilot teaching experiment investigating understanding of the single quantification by students in three sections of the first college level course Introduction to College Mathematics for Liberal Arts majors at the Hostos Community College of the City of New York in the Bronx. The college is the only bilingual (English/Spanish) institution of higher education in the CUNY public system, and the majority of students (65%) are ethnically Spanish speaking, while around additional 15 % are speakers of languages other than English and Spanish. Thus it was expected that we might observe not only impact of the English natural register upon the meaning of mathematical statements but the impact of Spanish or French language as well.

The course introduces students to elementary notions of Set Theory, Logic, Probability and Numerical Patterns. Each instructor has the freedom to design their own course syllabus, composed of four specialized domains, as well as the content of all tests and the final exam.

The aim of this presentation is twofold: on one hand, it is to complete the assessment of the learning process for both quantifiers, and on the other, to demonstrate a new approach to a clinical student interview, called the teaching-research student interview, which of course, has a twofold nature as well: to inquire and to teach. The interviews are teaching-research interviews because they fulfil two roles, that of investigation and that of teaching. They are also named in the sequel as "investigative teaching".

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL. 5, N. 3
October 2012

The Theoretical Framework and Methodology.

The study has been conducted within the theoretical framework of Teaching-Research NYCity model (Czarnocha, B. 2002, Czarnocha and Prabhu, 2006) utilizing cyclical investigations for the design of classroom intervention, its implementation, collection and analysis of data and the refinement for the next iteration of the intervention.

The research questions of the study were:

What are students' difficulties in negation and understanding of the quantified statements in the bilingual context?

What are the routes of improvement for students' understanding and mastery of the negation of quantified statements?

Each cycle of the teaching experiment in TR NYCity methodology thus plays a dual role, that of an inquiry and that of inquiry-based improvement of learning. TR NYCity model develops adaptive methods of instruction (CPRE Report, 2011). The teaching experiment was conducted by two collaborating instructors whose teaching techniques were similar, except for the degree of symbolic notation employed. Instructor I_1 used a minimal degree of symbolism, relying on verbal explanations. Instructor I_2 taught the quantifiers topic with intensive use of symbolical notation. In the second test, after the students were introduced to using universal quantifier $\forall$ and existential quantifier $\exists$, instructor I_2 asked them to first symbolize the original statements, then write the negation of the symbolized version, and ultimately translate the symbolized negations back to English. As an example:

Original sentence: "All men are mortal."
Symbolize: $(\forall x)M(x)$
Negation: $\neg(\forall x)M(x)$
Move the negation to inside the scope of the quantifier: $(\exists x)\neg M(x)$
Translate the negation into English: Some men are not mortal.

This difference in teaching was used to investigate the differences in effectiveness of each technique on student learning of quantifiers. Whereas together both classes had N=112 students, only 43 completed the pre-test and post-tests components of the data. The pre-test contained 4 propositions to negate (Statements 1,3,5,7) to assess student initial knowledge of the subject. Instruction of the quantifiers and their negations followed the assessment and, after this instructional intervention, a post-test was administered (Appendix). Each instructor taught the topic according to individual preference. The essential difference was the degree of introduced abstract logical notation.

Table Of Pre/Post Tests Results. (Pr/Po)

	Propositions to transform	I_1, Pr/Po N=32, + %	I_2 Pr/Po N=17, + %
1	Negation: All men are mortal.	14/75	37/100
2	Equivalence: It is not the case that all men are mortal.	34	18
3	Negation: All integers are whole numbers.	0/84	44/88
4	Equivalence: It is not the case that all integers are whole numbers.	28	36

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL. 5, N. 3
October 2012

5	Negation: Some athletes win gold medals.	11/16	44/88
6	Equivalence: It is not the case that some athletes win Gold medal.	19	36
7	Negation: Some numbers are not even.	14/63	30/88
8	Equivalence: It is not the case that some numbers are even.	75	27

Data is from two instructors (N_1 =32,N_2=17); …/… indicates the percentage of correct responses on the pre test and on the post test. It is clear from the pre-test scores that the level of initial preparation for the concept was quite different in both cohorts. One of the common challenges in both student cohorts are the difficulties in finding the equivalent statements to the negated statements of the type: "It is not the case that…". Interviews demonstrate that the difficulty was centered on understanding that the phrase means negation of the proposition. Once this was understood, the problems diminished. The results are ambiguous on the negation of the existential quantifier; on one hand I_1 has different results on the equivalence " It is not the case that some number are even." than on other items with existential quantifier. On the other hand, I_2 has strong positive results for negation of the existential quantifier, but not on the corresponding equivalent statements. It is clear from the pre-test scores that the level of initial preparation for the concept was quite different in both cohorts. To obtain better insight into students' problems, we turn to analysis of the interviews conducted from a sampling of students. Students were asked to come to the interview on the basis of the types of errors they made on the written test. The interviews were semi-structured. Each student was asked to provide the answer with explanation to at least one universal and one existential quantifier question. The interviews were conducted before a Smart Board, which videotaped the writings on the board together with the accompanying audio, without the images of the student or instructor. This set up satisfies the requirements of IRB. The recorded interviews are available to the public on request. The interviews were divided into two sections: first, a diagnostic section, which ascertained the knowledge of the student, then a didactic section, addressing the diagnosed weaknesses of the student. The teaching strategies discovered during the interviews become part of the instruction for the next iteration of this teaching experiment next semester.

Analysis of the Interview.

From this example of investigative teaching, a missing aspect to student understanding was discovered during the interview. Once identified, the incorporation of this missing aspect immediately helped the student to understand different cases.

1 *Instructor*: What does this proposition mean for you, "It is not the case that all men are mortal"
2 *Student*: "Not all men are mortal."
3 *Instructor*: Makes sense. What does it mean for you: "Not all men are mortal"?
4 *Student*: It means that "some men are mortal and some men are not mortal."
5 *Instructor*: And which of those two is the negation of "All men are mortal"?
6 *Student:* "Some men are mortal."

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL. 5, N. 3
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At the same time student starts introducing the variable P symbolizing a proposition and writes ~(~P) saying that therefore it's P.

7 *Instructor*: OK. What is P? Is it "All men are mortal?"

8 *Student:* Yes.

9 *Instructor*: So what we have in front of us is we have this (he circles "It is not the case..") which is a negation ~, and this (he circles "All men are mortal") is our P, proposition. So we have ~(all men are mortal).

10 *Student*: (writes under the text) Some men are not mortal.

11 *Instructor*: What made you so certain suddenly?

12 *Student:* Once you put a symbol like there, then it's become clear. Before I tried to figure it from the words, but here, once you have a formula…

Two issues are revealed in the above fragment:
Issue 1: Lines (5, 6) reveal a frequently encountered conceptual misunderstanding in perceiving "Some men are mortal" as the negation of "All men are mortal". The issue starts in the line (3,4) when the meaning of "Not all men are mortal" is understood as "Some men are mortal and some men are not mortal". It is intriguing that having this conjunction, the student chooses the first conjunct, not the second as the negation. This indicates an important cognitive issue. If they choose "Some men are mortal" as the negation, they do not see that if "All men are mortal" is true then "Some men are mortal must be true too, and therefore it can't be the negation of "All men are mortal." In the fragment above, the instructor attempted to direct student attention towards that contradiction (line 5), but the student took another route.
Issue 2: Lines (9 - 12) reveal that student understanding lacked the intermediate step between the verbal and symbolic levels of mathematical expression, which requires the symbol of negation of the proposition "All men are mortal" in front of it. This hypothesis was rapidly confirmed by the student's response to the next question:

13 Instructor: Find the equivalent to "It is not the case that some athletes win Gold medals."

14 Student: writes the following 3 lines:

It is not the case (that some athletes win Gold medals).
~ (that some athletes win Gold medal).
All athletes do not win Gold medal.

Precise coordination between verbal and symbolic levels will become part of next semester's iterated instruction of negation of quantifiers in our continuous cyclical teaching experiment.
The question concerning the role of formal notation instruction, which was pursued by I_2, has several possible answers. On one hand the quantitative data show its very strong and positive impact on students in terms of learning the negation of the existential quantifier, especially as compared with the results of instructor I_1. On the other hand the interviews show interesting differences between students, which indicate that some students were

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL. 5, N. 3
October 2012

matured enough to be introduced to the formal notations, others were not. The criterion to decide we take to be Vygotskyi's characterization: the spontaneous concepts of a student need to be just beneath scientific concepts in their development (Vygotsky, 1987) for the student to adapt easily to the scientific notation of the quantified statement. The student introduced symbolic notation by himself although incorrectly; it seems however that his error was related to the difficulty in understanding the phrase "It is not the case that.." as a single negation of the proposition (line 12). The readiness with which he picked up the procedure after the symbol of negation was introduced, indicate readiness of his own spontaneous concept to be formalized.

Impact of the natural language on student understanding

The claim of (Ferrari, 2004) and (Bardelli, 2011) that natural language has an impact on student understanding of logical statements has been confirmed in our investigations. The negation of "All men are mortal" as "Some men are mortal" (line 5, 6) is common among a large number of students. From the linguistic perspective, it is connected to negating the subject of the sentence. A second error encountered during the negation of "All men are mortal" is "All men are not mortal", where the negation of the verb occurs. The impact of natural language discussed in (Bardelle, 2011) generates both errors. Our hypothesis for the organization of the pedagogy in the next cycle iteration of the teaching experiment is: The <u>negation of the subject</u> AND <u>the verb</u> is the content of logical negation of the proposition, which of course distinguishes it from the negations encountered in a natural language.

An excellent example of the investigative teaching (teaching-research) in a bilingual (English/French) mathematical environment provides a very strong evidence for such an impact. The following series of instructor's questions and student responses led them to identify the source of difficulty in the incorrect translation of an corresponding phrase from French by the interviewed student:

1 *Instructor:* Find the equivalent statement to "It is not the case that some athletes win Gold medals."
2 *Student:* "All athletes win Gold medals."
3 *Instructor* (perplexed): Find the equivalent statement to "It is not the case that some numbers are even."
4 *Student:* "All numbers are not even."
5 Instructor (intrigued): Find the equivalent statement to "It is not the case that some athletes are winners."
6 *Student:* "All athletes are not winners"
7 *Instructor* (certain of formed hypothesis): Find the equivalent statement to "It is not the case that some athletes are winners."
8 *Student*: "All athletes win Gold medals".

After the last exchange it became clear that the issue is in the grammar, and on closer inspection it turned out that the issue was caused by inappropriate translation from French. The full transcript of the discovery will be presented at the conference.

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL. 5, N. 3
October 2012

Conclusions.

Our central aim was to investigate two research questions: the first inquires into the state of student knowledge vis-à-vis negation of both quantifiers, and the second inquires into the routes of improvement of that knowledge, once the need for improvement had been identified.

State of student knowledge the pretest demonstrates that students enter the subject strongly impacted by their intuitive knowledge of the logical negation of the proposition as "verb negation". [Some integers are whole numbers]→ [Some integers are not whole numbers]. That bias changes a significant degree upon the instruction. The interviews reveal that student understanding has misconceptions of which careful elimination is necessary to reach complete understanding.

Discovery of the routes for improvement takes place primarily during the individual interviews with students. Two strategies had been discovered: creating the verbal symbolic notation through which the transition to the symbolic one can be made, and using one particular structure "It is not the case that …" as the probe in the clarification of the meaning of negation.

Appendix: The instrument of assessment

Choose correct answer to each question.

NOTE: There are TWO different INSTRUCTIONS among the problems:

Find the negation or find the equivalent:

1A) Which of the following sentences is the negation of **"All men are mortal"**?

a. No man is mortal.
b. Some men are mortal.
c. Some men are not mortal.

1B) Which of the following sentences is equivalent to **"It is not the case that all men are mortal"**?

a. Some men are mortal.
b. Some men are not mortal.
c. No men are mortal.

2A) Which of the following sentences is the negation of the sentence: **"All integers are whole numbers"**?

a. Some integers are not whole numbers.
b. Some integers are whole numbers.
c. No integers are whole numbers.

2B) Which of the following sentences is equivalent to **"It is not the case that all integers are whole numbers"**?

a. Some integers are not whole numbers.
b. Some integers are whole numbers.
c. No integers are whole numbers.

3A) Which of the following sentences is the negation of the sentence: **"Some athletes win Gold medals"**?

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL. 5, N. 3
October 2012

a. No athlete wins Gold medal.
b. Some athletes do not win Gold medal
c. All athletes do win Gold medals.

3B) Which of the following sentences is equivalent to the sentence: **"It is not the case that some athletes win Gold medal"**?

a. No athletes win Gold medals
b. Some athletes do not win Gold medals.
c. All athletes win Gold medals

4) Which of the following sentences is the negation of the sentence: **"Some numbers are not even"**?

a. All numbers are not even
b. All numbers are even
c. Some numbers are even.

5) Which of the following sentences are equivalent to the sentence:" **It is not the case that some numbers are even"?**

a. Some numbers are even
b. All numbers are even.
c. All numbers are not even

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