LEARNING TRAJECTORIES FROM THE ARITHMETIC/ALGEBRA DIVIDE

Bronislaw Czarnocha
Hostos Community College,
bczarnocha@hostos.cuny.edu

William Baker	Olen Dias	Vrunda Prabhu
Hostos Community College	Hostos Community College	Bronx Community College
wbaker@hostos.cuny.edu	odias@hostos.cuny.edu	vrunda.prabhu@bcc.cuny.edu

Two hypothetical learning trajectories from the arithmetic-algebra divide are presented. Rational Number as the Gateway to Algebra trajectory has been developed through the Teaching-Research NYCity model (Czarnocha Prabhu, 2006) in mathematics classrooms of Hostos Community College and Bronx Community College of the City University of New York. Linear Equations trajectory has been designed on the basis of the craft knowledge of mathematics instructors and of the work done by Confrey et al, (2010) presenting posters of conjectured trajectories along the Common Core Standards of Mathematics. Both trajectories are scheduled to undergo research- based iterated development in the context of Teaching Experiment NYCity Model.

Keywords: algebra and algebraic thinking, high school education, CCSM Standards

Learning Trajectories.

The concept of a Learning Trajectory has recently acquired new importance as the organizing principle for the new Common Core Standards in Mathematics (CPRE, 2011). There are several definitions of a learning trajectory within the research profession; for the purpose of this work we take the definition of Clements, D. and Sarama, J., (2009):A **Learning Trajectory** (LT) of a particular mathematical concept consists of three components:

- a specific mathematical goal,
- a developmental path along which students' thinking and comprehension develops and
- a set of instructional activities that help students move along that path.

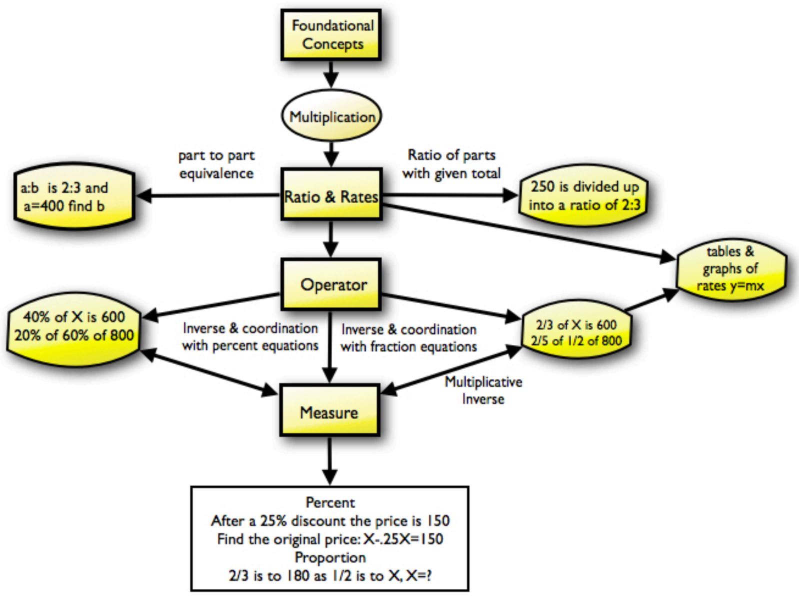
The idea of LT's has a wide range of applications. It can be an excellent assessment tool precisely informing the teacher about the successful pathways of mathematical thinking of students as well as about its weaknesses; at the same time it can be a tool, a map or a guide that informs the teacher about possible strategies for improvement of learning. Whereas there has been a systematic and continuous development of learning trajectory in elementary and middle school mathematics up to partitioning (Confrey, 2008) and fractions (Pettit et al 2010), there are no research-based trajectories in school algebra as yet, although certain hypothetical learning trajectories do exist (e.g. Learning Trajectory Displays of CCSM, K-12 by Confrey, Maloney and Nguyen, 2010). One of the critical questions to answer approaching research on learning trajectories in algebra is where to start. The answer depends on local conditions. In particular, in community colleges of the Bronx the most critical transition in learning mathematics is the transition between arithmetic and algebra. Akst (2005) asserts that of all students registering in remedial arithmetic courses in college, only 35% are passing subsequent course in algebra, the gateway out of remediation. Therefore, the transition between arithmetic and algebra requires a special attention that can be afforded only by the level of “micro-conceptual development” (Baroody, 2009) allowed by detailed learning trajectories for relevant concepts.

The Mathematics Teaching-Research group at Hostos CC and Bronx CC of CUNY have developed two LT’s in this critical area to assist students cross the ‘cognitive gap’ between arithmetical and algebraic thought: (1) Rational Number Sense as the Gateway to Algebra, and (2) Linear Equations trajectory. This second trajectory incorporates three fundamental meanings of the concept of the variable: variable as a Specific Unknown, variable as a General Number and variable in the Functional Relationship (Ursini and Trigueros, 2009).

Rational Number Sense as the Gateway to Algebra.

Fractions are often viewed as the foundational cornerstone for proportional reasoning which itself was viewed by Piaget as the beginning of formal mathematical thought and thus, by many as a foundation for algebraic reasoning. The rational number sense LT traces out students’ transition from arithmetic to algebra through their ability to understand multiple representations put forth by in Behr et al. (1993) extending the model due to Kieren, which views the fraction concept as composed of five related sub-constructs, the primary one of part-whole and four related fraction sub-constructs: ratio, operator, quotient and measure. This view of fractions has been used as a foundation for rational number sense (Lamon,2007) and as the basis for the rational number sense trajectory (Confrey et al, 2009). It was used to map out student’s concept schema as they transition from arithmetic to algebra (Baker et al, 2007). This schema has been developed through teaching research experiments, in which students’ results were analyzed to understand rational number sense and the transition to proportional and algebraic thought i.e. how students cross the cognitive gap from arithmetic to algebra. A lesson plan that follows this schema would build upon students basic concept knowledge of part-whole and proportional thinking on an informal level as well as the procedural knowledge of multiplication to first, develop and extend ratio and rates to algebra. This would include proportion problems, rates and perhaps graphing rate formulas such as $d=rt$ or $y=mx$ on the number line. Second, the operator concept would be used through inverse and coordinated processes to introduce fractional and percent equations such as $0.25X=120$. Finally, the two-sided number line along with algebra would be used to solve percent problems such as the discount problem shown. Proportion problems with complex fractions and decimals would also be set up.

Figure 1: Rational Number Sense as a Gateway to Algebra



The APOS or action-process-object-schema learning theory is the theoretical foundation for the learning trajectory that begins with variable as unknown (figure 2). The APOS model was used by Czarnocha and Prabhu, as members of the Research in Undergraduate Mathematics Community (RUMEC) to describe concept formation as a cycle beginning with actions on existing concepts, that are interiorized into processes and with further generalization into objects (Czarnocha et al,1999). Process/object models are useful for the assessment of student thinking along the arithmetic/algebra divide, which has been described by (Sfard 1991) as a transition from "operational thinking" where algebraic expressions such as 2x+ 3 have little meaning until the value of x is known, to "structural thinking" in which such an expression can be combined with other expressions and manipulated to obtain a goal (Tall et. al., 2000). The arrows from the operator and measure constructs, utilizes the 'reversal' and 'coordination' of arithmetical processes to 'interiorize' (Dubinsky,1991) students' operational to a more structural understanding. These critical processes of reversal, coordination and interiorization and types of 'reflective abstraction' (Dubinsky,1991) which transition students along the APOS continuum and represents a critical component of this learning trajectory. As noted in figure 2, language can be a useful mediating factor in the APOS continuum (Baker & Czarnocha, 2002) i.e. algebraic thought can be motivated by asking students to find the number such that 5 more than the number is 12. B. Czarnocha and the TR team have used the APOS model in the teaching and learning of calculus (Clark et al 1997) and (Czarnocha et al 2000) they have applied process/object theories to development of learning material in the transition from arithmetic to algebra (Baker & Czarnocha, 2008) and as the basis of the learning material on a CUNY-College Collaborative Incentive Research Grant (2010-2011)

An Instance of the Adaptive Instruction: Linear Equations Trajectory.

The success of the LT approach depends on the implementation of the associated adaptive pedagogy, whose basic premise is adjusting to student learning needs, thereby creating a better and more effective experience for the learner. "*For that to happen, teachers are going to have to find ways to attend more closely and regularly to each of their students during instruction to determine where they are in their progress toward meeting the standards, and the kinds of problems they might be having along the way. Then teachers must use that information to decide what to do to help each student continue to progress, to provide students with feedback, and help them overcome their particular problems to get back on a path toward success.* This is what is known as adaptive instruction and it is what practice must look like in a standards-based system." Consortium for Policy Research in Education Report (CPRE, 2011). This report refers to the Common Core Standards in Mathematics (CCSM).

Mathematical situation. The teacher in the classroom observed student difficulties with the problem: *Solve for y in terms of x: 3x – 2y = 6* .The difficulties were observed during the pre-final exam.

The most typical student error was the following:

3x - 2y = 6
<u>-3x</u>
-2y = 6
y = -3

Upon positioning the difficulty on the concept map below the teacher can relate the issues to the remaining concepts of the trajectory and design a Supplementary Learning Trajectory (SLT) geared to the needs of students.

Teaching-research Diagnosis. (a) absence of awareness of the functional relationship between x,y variables evidenced by transforming the problem to a solution of a simple one unknown equation leading to (b) misapplication of the variable as specific unknown, (c) the absence of understanding the algebraic meaning of the equality symbol "=" evidenced by adding -3x to one side of the equation only. Generally, one observes lack of structural understanding of linear equations with two variables. In particular, the students' sense of equality becomes unstable with addition of the second variable.

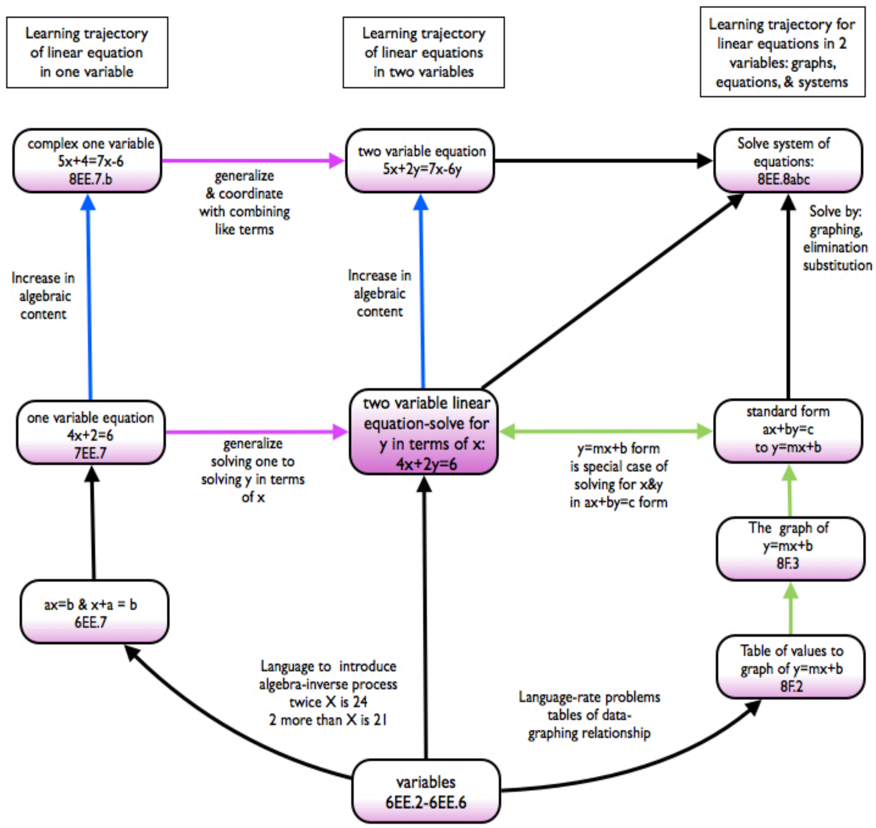


Figure 2: An Instant of Adaptive Instruction. Linear Equations Trajectory

Three possible SLTs are indicated on the concept map. The pink one leads along the process of generalization from a formally similar equation in one variable to a corresponding equation in two variables. This trajectory is useful if the class has mastered solving simple one variable equations. Otherwise, the second trajectory (green) is available via the graphing component of the schema, which connects the challenge of the problem with its foundations within the concept of a variable, meaning of equality and the functional relationship between x & y. The cognitive fragility of the left upper rectangle in the concept map is known in the literature. Filloly & Trojano (1989) and Ursini & Trigueros (2009) observe that the increase of algebraic content is a serious problem for students because its solution departs from the equation e.g. 4x + 2 = 6. The two horizontal pathways indicate abstraction from and generalization of one variable equation to two variables – an arduous process which according to many investigations focused on problems that students have with generalization as they begin to study algebra in middle school. Most conclude that generalization is a difficult obstacle for the majority of these students (Bell & Mallone,1993; Azarello et al,1994; Bednarz et.al., 1994; and Bolea et al., 1998). The graphing trajectory (green), develops the concept of "Solve for y in terms of x" through transformation of a standard form of an equation into known functional relationship y=mx+b.

The third trajectory, joins the concept of the variable to the discovered difficulty along the theme of algebraic equality " =" through a series of "scale balance" type of problems. The assumed equilibrium of the scale in such problems is the metaphor for algebraic equality " =" The possibility of distinguishing three different learning progressions within the concept map demonstrates the versatility of such an integrated concept map/learning trajectory for classroom teachers and its usefulness in addressing diverse learners. According to (Ursini and Trigueros, 2009), the best, flexible development of the schema of the variable is to engage in coordination of three subschema: (1) variable as a specific unknown, (2) variable as general number, (3) variable in a functional relationship, what implies the use of all three trajectories, because all three subschema are involved in the problem.

Teaching Sequence of Mathematical Activities which are to propel a student along the pink trajectory of generalization.

The trajectory uses writing mathematics approach to increase the meta-cognition and reflection upon the methods of solution.

Problem 1

Solve for X. As you solve write every step you make in the solution). Look at 3 descriptions, collect similar actions in 3 examples and write them as one instruction to all three problems.

$2x + 7 = 15$

$-4x + 8 = -28$

$5x – 3 = 12$.

My general description is…

Problem 2.

Look at the following 3 examples, which are similar but different from the previous series and solve for x by applying your general prescription to those two cases. Write carefully your steps and their order.

$2x + y = 15$

$-4x + y = -28$

$5x – y = 12$

Problem 3

Solve for y by applying your general description to those two cases. Write carefully your steps and their order.

$2x + y = 15$

$-4x + y = -28$

$5x – y = 12$

Write the general description of steps for the instruction :" Solve for y in terms of x" looking at the last two problems.

Problem 4

Solve for y in terms of x:

$4x + 2y = 12$

$6x - 3y = 15$

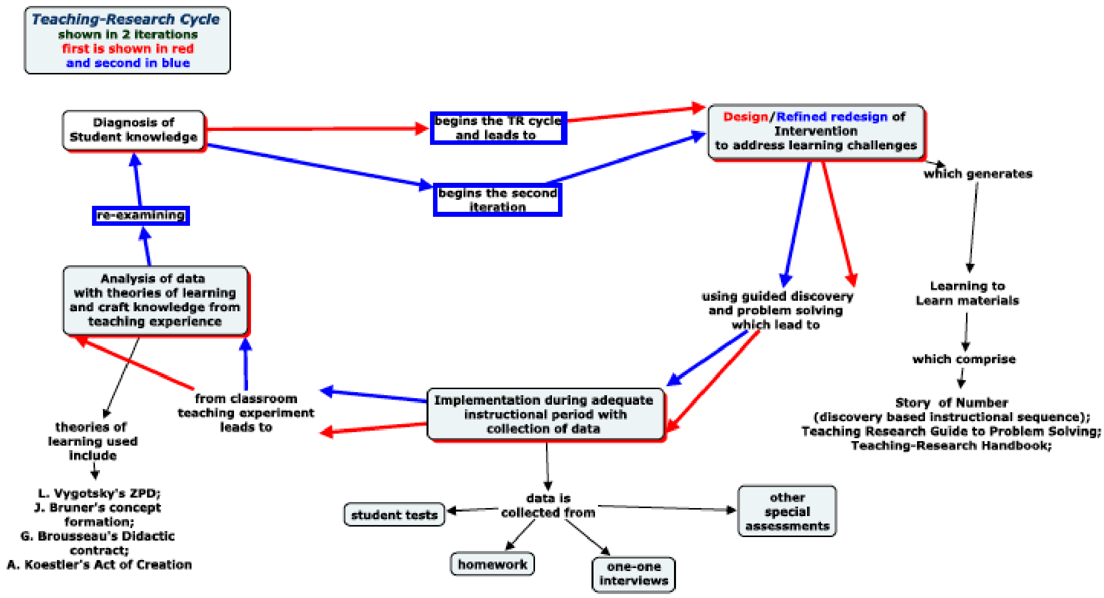
$-2x + 3x =15$

$-4x + 2y = 15$

What is the critical computational difference between the last two and the first two examples?

Both trajectories are scheduled to be investigated and refined through the cyclical Teaching Experiment NYCity model (Czarnocha and Prabhu,2006) conducted in the classrooms of remedial arithmetic and algebra at the Hostos CC and Bronx CC of CUNY. The data from the next cycle will be ready for the NA PME 2012 conference.

Fig. 3. Cycles of work by TR/NYCity model.



REFERENCES

Akst, G., (2005) Performance in selected mathematics courses at the City University of New York: Implication for retention, Office of Institutional research and Assessment.

Arzarello,F., Bazzini, L,& Chiappini, C. (1994).The process of naming in algebraic problem Solving. In J.P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.40–47).Portugal:University of Lisbon.

Baker,W**.**, Dias,O., Doyle, K., Czarnocha, B., Prabhu, V., (2009) A Study of Adult Students Learning Fractions at a Community College. *Annales of Polish Mathematical Society, 5th Series, Didactica Mathematicae 32*, pp. 5 – 48, Krakow, Poland.

Baker,W., Czarnocha, B., (2008) The transition from Pre-algebra to algebra: Learning Theories Based Instructional tasks. Mathematics Teaching-Research Journal on line, vol.3 (1), pp.35-47.

Baker, W., Czarnocha,B.,Dias,O.,Vrabu,V.,(2010-2011) Community College Collaborative Incentive Research Grant Program, CUNY, Problem Solving in Remedial Mathematics: A Jump Start to Reform

Baker W., Czarnocha, B. "*Written Metacognition and Procedural Knowledge,"* Proceedings of the 2nd International Conference on the Teaching of Mathematics, University of Crete, Hersonissos Crete, Greece, 1-6 July 2002. John Wiley & Sons Inc.

Baroody, A.J., Baroody, A., Lait, M., Li,X., (2009) Young Children's Understanding and Application of Subtraction-Related Principles Mathematical Thinking and Learning (11),1&2, pp.41-60

B Behr, M.J., Harel, G., Post, T. and Lesh, R.: 1993, 'Rational numbers: Toward a semantic analysis-emphasis on the operator construct', in T.P. Carpenter, E. Fennema and T.A. Romberg (eds.), Rational Numbers: An Integration of Research, Lawrence Erlbaum Associates, New Jersey, pp. 13–47.

Bednarz, N., Janvier, B. (1994). The emergence and development of algebra in a problem solving context. PME 18, Portugal, 2, 64-71.

Bell, A. et Malone, J. (1993). Learning the language of algebra. *In* I. Hyrabayashi, N. Nohda, K. Shigematsu et E-L. Lin (dir.), *Proceedings of the 17thInternational Conference for the Psychology of Mathematics Education* (Vol. 1, p. 130-137). Tsukuba, Ibaraki, Japon: University of Tsukuba.

Bolea, P., Bosch, M., Gascon, P. (1998) The role of algebraisation in the study of a mathematical organization. *Proceedings of CERME 1.*

Clements,D. and Sarama, J.(2009) *Learning and Teaching early Math:The Learning Trajectories Approach*. New York: Routledge.

Clark,J., Cordero,F., Cottrill,J., Czarnocha, B., DeVries,D. St. John,D., Tolias,G., Vidakovic,D. (1997) Constructing a schema: The case of the chain rule*, Journal of Mathematical Behavior*, **16**(4), 345-364,

Confrey,J., Maloney, A., (2010), The construction, refinement and early validation of the equipatitioning learning trajectory, ICLS, 1, pp.968-975.

Confrey,J., & Maloney,A.P., (2009) Linking Mathematics to Learning Trajectories: A Case study from North Carolina, 2008-2009, In F. Mosher, J. Confrey & P. Daro (Eds).: New York, N.Y.: Columbia University Press.

Confrey (2008) A Synthesis of the research on rational number reasoning: A learning progressions approach to synthesis. Paper presented at the 11th international congress of mathematics instruction, Monterrey, Mexico.

Czarnocha,B., Prabhu, V. (2006) Teaching-Research NYCity model, *Dydaktyka Matematyki*, vol.29, Krakow, Poland.

Czarnocha, B., Dubinsky,E., Prabu,V.,Viadokovic,D.,(1999) *One theoretical perspective in undergraduate mathematics education research.* In O. Zaslavsky (Ed.) Proceedings of the 23rd Conference of PME, Haifa, Israel,1,95-110

Dubinsky, E., (1991) *Reflective Abstraction in Advanced Mathematical Thinking*, in Advanced Mathematical Thinking (D. Tall, ed.), Kluwer, 95-12.

Filloy, E. et Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics,* .9(2), 19-25.

Petit M, Laird R and Marsden E. (2010) A focus on Fractions: Bringing Research to the Classroom. Taylor & Francis Group, New York.

Sfard, A. (1991) On the dual nature of mathematical conceptions: reflections on processes and objects as two sides of the same coin. Educational Studies in Mathematics 22 p.1-36

Tall, D., Thomas, M., Davis, G., Gray, E., Simpson, A. (2000) What Is the Object of the Encapsulation of a Process? Journal of Mathematical Behavior, 18 (2), 223-241

Ursini, S. & Trigueros, M. (2009). The Role of Variable in Elementary Algebra: An Approach through the 3UV Model (Ed.) Roberta V. Nata , in Progress in Education, Volume 19 Nova Science Publishers, Inc.

CPRE http://cpre.wceruw.org/

The Common Core State Standards (CPRE, 2011) http://www.corestandards.org/