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Reflection on points

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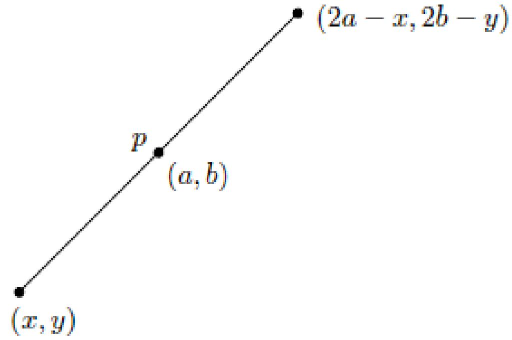

FIGURE 1. $\sigma_p[(x,y)] = (2a-x, 2b-y)$

introduction:

When visiting a Math and Science fair as a judge, the author of this article was inspired by a student's project. That project was about satellite and transformation of signals between satellites. This article is developed from that project. Another reason for this article is to show that how easy and beautiful it is to present elementary ideas of Euclidean geometry and after two thousand years still there are new theorems discovered in this field.

Without any further interruption, we present the theorem:

Theorem 1. *If p is a point in the plane, we denote by $\sigma_p : \mathbf{R^2} \to \mathbf{R^2}$, the reflection around the point p. If $p_1, p_2, \ldots p_{2n+1}$ are any $2n+1$ points in the plane (here n is any positive integer), then*

$$[\sigma_{p_{2n+1}} \sigma_{p_{2n}} \ldots \sigma_{p_2} \sigma_{p_1}]^2 = id$$

Here id stands for the identity map of the plane. Note that we must have odd number of points for this theorem to be true and the theorem does not hold for even number of points. The reason is going to be clear from the proof.

The proof of this theorem is very simple. To make it more simpler, we first give the following lemma:

Lemma: Let p has the coordinate (a,b). Then

$$\sigma_p[(x,y)] = (2a-x, 2b-y)$$

See the figure 1. Let $\sigma_p[(x,y)] = (s,t)$. We need to find s and t. Since (a,b) is the midpoint of (x,y) and (s,t), we must have

$$\frac{x+s}{2} = a \Rightarrow s = 2a - x$$

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and

$$\frac{y+t}{2} = b \Rightarrow t = 2b - y$$

Corollary: Let $p_1, p_2, \dots p_k$ be k points on the plane having coordinate $(a_1, b_1), (a_2, b_2), \dots (a_k, b_k)$ respectively. Then applying the above lemma k times, we have

$$\sigma_{p_k}\sigma_{p_{k-1}} \dots \sigma_{p_2}\sigma_{p_1}[(x,y)] =$$

$$(2a_k - 2a_{k-1} + \dots (-1)^{k-1}2a_1 + (-1)^k x, 2b_k - 2b_{k-1} + \dots (-1)^{k-1}2b_1 + (-1)^k y)$$

Proof of the main theorem: Let k is odd, say $k = 2n+1$, and we are given $2n+1$ points in the plane, say $p_1, p_2, \dots p_{2n+1}$. The coordinates of these points are $(a_1, b_1), (a_2, b_2), \dots (a_{2n+1}, b_{2n+1})$ respectively. For simplicity, let us denote:

$$T_{2n+1} = 2a_{2n+1} - 2a_{2n} + \dots (-1)^{2n+1-1}2a_1 = 2a_{2n+1} - 2a_{2n} + \dots 2a_1$$

and

$$S_{2n+1} = 2b_{2n+1} - 2b_{2n} + \dots (-1)^{2n+1-1}2b_1 = 2b_{2n+1} - 2b_{2n} + \dots 2b_1$$

Then by the lemma and its corollary it easily follows that

$$[\sigma_{p_{2n+1}}\sigma_{p_{2n}} \dots \sigma_{p_2}\sigma_{p_1}]^2(x,y) =$$

$$[\sigma_{p_{2n+1}}\sigma_{p_{2n}} \dots \sigma_{p_2}\sigma_{p_1}][\sigma_{p_{2n+1}}\sigma_{p_{2n}} \dots \sigma_{p_2}\sigma_{p_1}](x,y) =$$

$$[\sigma_{p_{2n+1}}\sigma_{p_{2n}} \dots \sigma_{p_2}\sigma_{p_1}](T_{2n+1}-x, S_{2n+1}-y) = (T_{2n+1}-(T_{2n+1}-x), S_{2n+1}-(S_{2n+1}-y)) = (x,y)$$

Thus $[\sigma_{p_{2n+1}}\sigma_{p_{2n}} \dots \sigma_{p_2}\sigma_{p_1}]^2 = id$.
End of the proof of the main theorem.
From the above proof, we can also see that what goes wrong if k is even. In the case k is even, say $k = 2n$, then in the above proof:

$$T_{2n} = 2a_{2n} - 2a_{2n-1} + \dots (-1)^{2n-1}2a_1 = 2a_{2n} - 2a_{2n-1} + \dots - 2a_1$$

and

$$S_{2n} = 2b_{2n} - 2b_{2n-1} + \dots (-1)^{2n-1}2b_1 = 2b_{2n} - 2b_{2n-1} + \dots - 2b_1$$

And so

$$[\sigma_{p_{2n}}\sigma_{p_{2n-1}} \dots \sigma_{p_2}\sigma_{p_1}]^2(x,y) =$$

$$[\sigma_{p_{2n}}\sigma_{p_{2n-1}} \dots \sigma_{p_2}\sigma_{p_1}][\sigma_{p_{2n}}\sigma_{p_{2n-1}} \dots \sigma_{p_2}\sigma_{p_1}](x,y) =$$

$$[\sigma_{p_{2n}}\sigma_{p_{2n-1}} \dots \sigma_{p_2}\sigma_{p_1}](T_{2n}+x, S_{2n}+y) = (T_{2n}+(T_{2n}+x), S_{2n}+(S_{2n}+y)) = (2T_{2n}+x, 2S_{2n}+y)$$

So we do not get identity in general unless, of course, $T_{2n} = 0$ and $S_{2n} = 0$ which is not true in general. We end this article with a illustration of this theorem for $k = 7$. See figure 2 below. Here we denote $p_1, p_2 \dots$ etc by $A, B, C \dots$ and we start with the point 1 and when we reflect 1 about the point A, we denoted the reflected point

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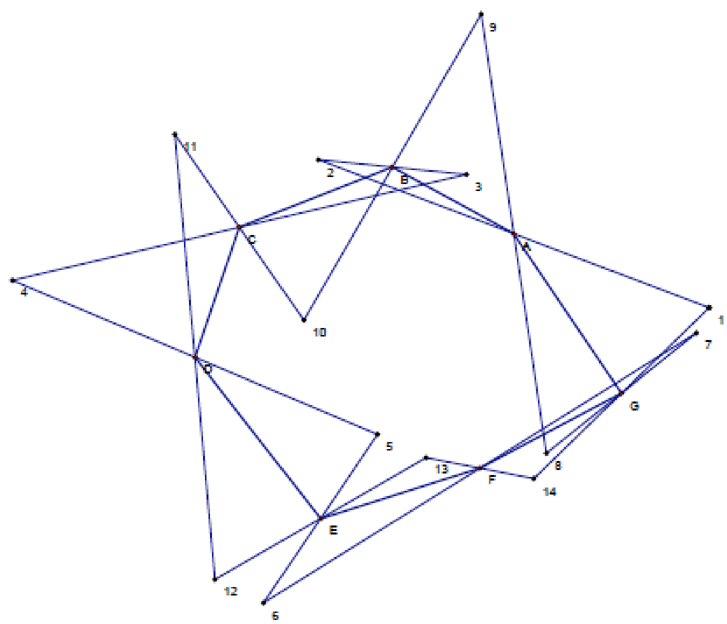

FIGURE 2. The above figure illustrate the theorem for seven points. That is if you reflect any point on the plane about seven points cyclically two times, you come back to the starting point.

by 2, when we reflect the point 2 about the point B, we denoted the reflected point by 3 and so on.

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