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Proportional Reasoning and Polya's Problem Solving in Pre-Algebra Mathematics

Ahmad Ibraheem, Kathleen M. Doyle, Bronislaw Czarnocha, and William Baker

Hostos Community College

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Abstract

Problem solving is one of the most important and difficult areas of teaching and learning mathematics. Curriculum reform efforts in secondary school and college level mathematics stress the need for educators and students to focus on the process of problem solving rather than the result. In this study, students enrolled in a remedial pre-algebra course at a community college were evaluated on their performance in each of Polya's four stages of problem solving as they attempted to answer an abstract problem involving proportional reasoning. The results of this study reveal that proportional reasoning is shown to be strongly influenced by reading for comprehension, computational proficiency, and metacognitive reflection. The results of study also demonstrate that students' scores approximate a standard normal distribution only in stage three, the computational stage, which is modeled most frequently in classroom presentations and textbooks, and reveals deep lack of students' familiarity and exposure to the other three stages in their education experience.

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Proportional Reasoning and Polya's Problem Solving in Pre-Algebra Mathematics

Most mathematics educators would agree that many students have difficulty with problem solving, especially weaker students who often enroll in remedial mathematics courses at two-year or community colleges. In *Beyond the Crossroads*, the attitude these students have toward problem solving is eloquently stated, "for many students, mathematics is viewed as a string of procedures to be memorized, where right answers count more than right thinking" (Blair, 2006). To encourage students to become independent problem-solvers educational reformers in mathematics suggest that teachers and instructors act more as a facilitator of students' thinking rather than simply modeling problem solving.

The most common method used to introduce problem solving is Polya's four stage method which emphasizes the entire process of problem solving rather than a narrow focus on how to obtain a solution. Polya proclaimed that by guiding students through all four stages of problem solving a mathematics teacher could help students develop a sense of independent thinking. In this study, a problem-solving worksheet based upon proportional reasoning was developed to investigate each of Polya's four stages: 1. Understanding the Problem, 2. Devising a Plan, 3. Carrying out the Plan and 4. Looking Back. A rubric was developed to score the results for adult students in pre-algebra mathematics. The first and second stages of Polya's method involve critical reading and thinking to determine the required strategy and the fourth involves reflecting upon the process. However, it is the third stage which receives the bulk of the attention in educational practice. This third stage is modeled in classrooms and textbooks as well

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as rubrics used for grading. The students' work was analyzed and used to reflect upon students' proficiency in each of the four stages of Polya's process, the relationship between the stages and implications for the pedagogy of teaching problem solving.

Theoretical Framework

While problem solving has always been an integral part of mathematics education, the present day study of problem solving owes much to the work of George Polya, "the father of the modern focus on problem solving in mathematics education" (Passmore,2007, p.44). In a similar vein, Schoenfeld (1987) writes of the impact of Polya's work, "for mathematics education and for world of problem solving [Polya's work] marked a line of demarcation between two eras, problem solving before and after Polya" (p. 27).

In the first stage of Polya' method, students must read and assimilate the information given to determine what they are asked to find and which information presented is relevant to this goal. The difficulties students have reading mathematical text for understanding is pointed out by Philips et al. (2009) who states that "it is important to realize that mathematics is a language all its own" (p.468). Fan and Zhu (2007) found that only 27.8% of the problems in nine U.S. secondary school mathematics textbooks contained solutions that modeled this first stage of Polya's methodology.

In the second stage of Polya's method, students must make a general plan and select relevant methods or appropriate heuristics for solving the problem. This stage involves relating the problem and its information to one's problem-solving schema (i.e.

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one's internal representation of similar experiences). A schema allows an individual to organize and recognize their thought processes. At this point, students who have an appropriate schemata or internal representation of similar experiences can synthesize this information into a strategy or plan. Xin (2007) notes that students who are successful at problem solving were those students who were able to identify the underlying structural information required to solve a problem. In contrast, students without an organized schema must consider each bit of information separately one at a time. Students who rely on a schema which has not been organized into a hierarchy of relationships are described by Sfard (1991) as being at a disadvantage since they are unable to assimilate new knowledge easily. Fan and Zhu (2007) found that this second stage was represented or modeled in only 20.2% of the problems.

In the third stage of Polya's method, students perform the computations required to implement the plan devised in the second stage, keeping on track to obtain the solution. Fan and Zhu (2007) found that this third stage was modeled by 100% of the problems and arguably, receives the bulk of the attention in the classroom when instructors model problem solving on the board. Taylor and McDonald (2007) confirm that this third stage receives the bulk of attention in university lectures and laboratory sessions.

In the fourth stage, students review what they have done, check the correctness of their solutions, and reflect on key ideas and processes in order to generalize or extend these processes and results. This stage involves what is frequently referred to as

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metacognition where students reflect and evaluate the thought processes that were used to solve a problem. This fourth stage was modeled in 43.2% of the problems in U.S. textbooks involved in the study by Fan and Zhu (2007).

Problem solving is typically distinguished from computation in mathematics by the additional requirement on the part of the problem solver to determine what operation, procedure or sequence of procedures needs to be applied to obtain a solution. In computation problems, the operation is given in the problem. The same is not true for word problems where the student must determine which operation to use in order to solve the problem. Thus one distinguishing factor in problem solving is that the strategy or algorithm to find the solution is not clear and in the face of this uncertainty the problem solvers have to rely upon their own reasoning skills and knowledge base.

While the unique aspect of problem solving is the element of an unknown path for the student, the traditional approach of instructors and the one most often emphasized in textbooks is to focus on the third or computational stage of Polya's method and to treat problem solving as one would instruction in computational proficiency. The futility of this approach is described by Lester (1983) "…good problem-solving behavior usually is not fostered by having students imitate how teachers solve problems. Because teachers typically demonstrate only correct moves, students often come to view problem solving as that of delving into a mysterious bag of tricks to which only a select few are privy" (p. 229).

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Remedial mathematics students have difficulty with problem solving because of their inability to measure their progress toward a solution and to switch between methods of solution. One approach advocated by reform mathematicians is to focus students' attention on the reasoning skills found in the second stage and to consider multiple solutions. The cognitive techniques of comparing, reflecting on and discussing multiple solution methods have been advocated by mathematics educators for over 20 years.

Method

In the work that follows remedial college students were given a problem that involves proportional reasoning. They were evaluated on each of the four stages of Polya's problem solving stages. In the first stage, we measured their ability to read and understand the information presented. In the second stage, students were asked to recognize multiple solutions. In the third task, students performed a proportional computation, and finally students were asked to reflect upon the problem structure. Their responses were scored using a rubric that emphasizes multiple methods of solutions. The results were used to analyze their proficiency in each of the four stages as well as the relationships of these stages to one another.

Research Questions

The first research question is: How do the students perform on each of four stages, what is the distribution of students' scores in each stage, and what is the correlation between stages?

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The second research is: To what extent do students' reasoning skill, as evaluated by their scores on the other three stages, determine or predict their score on the third computational stage?

The third question is: To what extent do students' scores on the other three stages determine or predict their score on the second stage?

Setting

The sample included 117 adult students taking developmental mathematics at Hostos Community College, an urban community colleges in the City University of New York system. Students were given worksheets that contained four exercises and informed that their participation was voluntary and would not impact their course grade. The student body at Hostos Community College is predominately comprised of females (80%) and minorities (95%). It is important to note that the student body used in this study is the mathematically weakest group of adult students applying to the CUNY system, those who failed both the algebra and pre-algebra placement exams in mathematics. The problem solving exercise chosen was used in two earlier studies. Charalambos and Pitta (2007) studied 600 5th and 6th grade students' scores in 50 exercises that evaluated understanding of different fractional concept (sub-constructs): ratio, part-whole partitioning and fractional equivalence was recorded along with the correlation between these concepts. It was determined that, part-whole partitioning with a mean score of (0.75), ratio, with a mean score of (0.64), and fractional equivalence, resulted in the following highly significant correlations.

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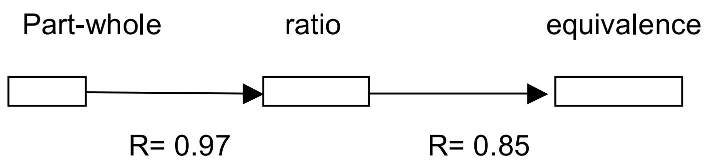
Figure 1 (excerpted from figure 2, Charalambos et. al., p.306, 2007)



When this study was duplicated with 95 adult students taking remedial pre-algebra mathematics at Hostos Community College by Czarnocha et al. (2009), the mean scores were similar to the results obtained with the childrens' part-whole (0.66) ratio (0.57) and fractional equivalence (0.58) however the correlation between ratio and part-whole (R=0.12) was not significant at the 0.05 level and neither was the correlation between ratio and fractional equivalence (R=0.05). The correlation between fractional equivalence (0.53) and part-whole was significant at the 0.01 level. These results suggest that the adult students did not understand the relationship between these proportional reasoning exercises and their fraction schemata and thus, this study was conducted to investigate the proportional reasoning schema of adult students. The worksheet was given to approximately ten sections of pre-algebra with the request that instructors leave 15-20 minutes for students to complete it at the end of class. The exercises and rubrics are included in Appendix A.

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Results

Research Question 1
Students' performance in each stage and their relationships

Table 1 Mean Score and Standard Deviation of the four stages

	Stage 1	Stage 2	Stage 3	Stage 4
Mean	2.42	1.77	2.22	1.64
St. Dev.	1.58	1.19	0.67	1.13
n	111	112	110	111

The students had the most difficulty with the reasoning skills of stage 2, and the metacognitive skills of looking back to assimilate what they had done in stage 4. These results were expected and support the characterization of developmental mathematics students as experiencing difficulty planning and monitoring their problem solving.

Table 2 Correlations between the stages

a i

		[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	P b	1	5	[illegible]	[illegible]
	[illegible]	.	0		
	N	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	P C	*	1	*	*
	[illegible]		.		
	N	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	P		*	1	
	[illegible]			.	
	N	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	P	*	*		1
	[illegible]				.
	N	[illegible]	[illegible]	[illegible]	1

* [illegible]

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The correlation between stages 1, 2 and 4 of Polya's method were all significant at the 0.01 level, however the only significant correlation involving stage 3 was that between stage 3 and stage 2.

This finding indicates that when stage 3 tasks are used to evaluate students' in mathematics then stages 1 and 4 of Polya's method will not have any significant impact upon their performance. While stage 2 will affect stage 3, its impact is limited to the square of the correlation coefficient $(0.265)^2 = 0.07$ or 7%. In contrast, stage 2 appears to correlate significantly with each of the other stages.

Figure 2 Graphical representations of student scores

The following is a graphical representation of student scores on the 4 stages of problem solving. The column charts show the exact distribution of scores and their line format to their right.

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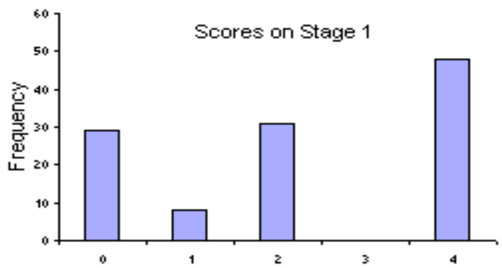

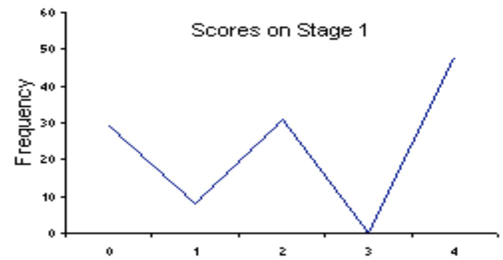

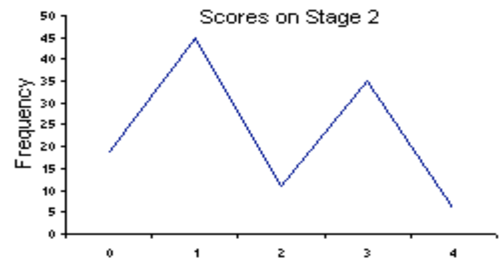

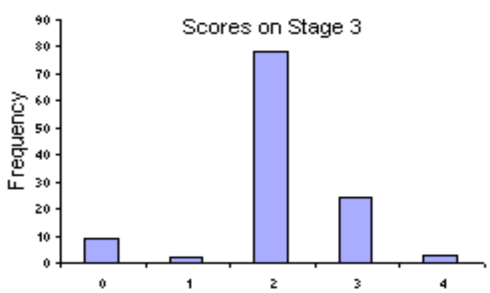

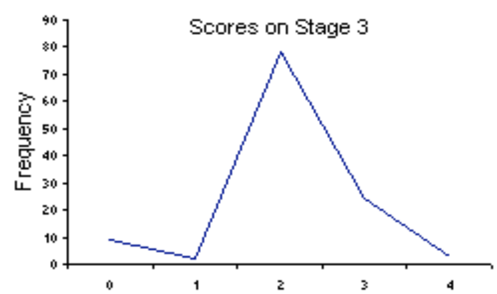

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When the results of figure 2 are compared with the work of Fan and Zhu (2007), there appears to be a relationship between the representation of each stage in textbooks and students' performance on that stage. The stages that are modeled most often in textbooks correspond to normally distributed student performance on that stage.

Table 3 Number of Problems whose solutions modeled Polya's problem-solving stages in the selected U.S. textbooks. (excerpted from table 3, Fan and Zhu p.67)

Stage in Polya's Method	Percent Representation in selected U.S. textbooks
Stage 1) Understanding the Problem	27.8%
Stage 2) Devising a Plan	20.2%
Stage 3) Carrying out the Plan	100%
Stage 4) Looking Back	43.2%

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The results in figure 2 show that stage 3 resembles a normal distribution and this is the stage that is modeled 100% in textbooks. Stage 4 receives the second most attention in textbooks and the representation of student responses is nearly a normal distribution. Stages 1 and 2 are the least represented in textbooks and students' performance on those stages does not approximate a normal distribution. The more often students were exposed to a certain stage of Polya's method in textbooks, the more likely they were to exhibit responses to the present research questions that were normally distributed.

Research Question #2

When stages 1, 2 and 4 were used as independent variables in a multivariate linear regression analysis to predict students' scores on the third stage only the coefficient of the second stage was statistically significant at the 0.05 level. This indicates that instruction in Polya's stage 2 is the best model for affecting stage 3. Thus, the amount that instruction at stage 2 can affect students' performance on stage 3 is $(0.265)^2 = 0.07$ or 7%.

Research Question #3

To what extent do critical reading, computational skill level and the ability to reflect upon the structure of the problem affect a student's ability to use and recognize multiple strategies?

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Table 4-A R-Correlation Value for Linear Regression Model

M	R			
1	[a]			0

a P S

Table 4-B Significance of Linear Regression Model

V [b]

M			d		F	
1	R	6	3	3	2	0
	R		0	2		
	T	4	4			

a P ,

b D

Table 4-C Significance of Coefficients in Linear Regression Model

a

M		d		i		
		B			t	S
1	(	9	8		2	8
	A	7	6	3	6	0
	A	3	4		8	
	A				2	

a D S

The R-value of the combined model was 0.612 thus approximately 37.5% of students' scores on stage 2 were predicted by the other three stages. This result indicates that students' ability for proportional reasoning is strongly influenced by a

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variety of skills: reading for comprehension, computational proficiency and metacognitive reflection.

Discussion of Results

The proportional reasoning exercise used in this study had previously been given to children (Charalambos & Pitta-Pantazi, 2007) as well as adults students in remedial pre-algebra classes (Czarnocha et al., 2009). These earlier results had indicated that children and adults were equivalent in mean scores or percentage correct when answering this problem. An important difference between the children and adults was that the children demonstrated highly significant correlations between these reasoning exercises and their conceptual part-whole knowledge of fractions as well as between these reasoning skills and their ability to perform procedural tasks of fractional equivalence. In contrast, the adult students did not, indicating a disconnect between their fractional schema knowledge and their proportional reasoning skills. This study is an attempt to explore the connections of their problem-solving schema by matching adult students' reasoning patterns with Polya's four stages and analyzing the relationships between them.

How well did students do on each stage? What is the distribution of each stage and what is the relationship between the stages? The results of table 1 indicate that adult remedial students have the most difficulty with the planning and reflection stages of Polya's method and the only stage in which the mean score was above a 2 was the understanding the problem stage. Thus these adult students on average demonstrated

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between a limited to a partial understanding of the structure of this problem and strategies required to solve it.

The results of table 3 and figure 2 indicate that stage 3, which receives the bulk of instruction time and textbook presentation, was closest to a normal distribution. The first and second stage in which students must decide what plan to select and receive the least amount of exposure in texts and class presentation are not even close to a normal distribution. The bimodal distribution exhibited in figure 2 for stage 2 suggests that some students are not familiar with the planning processes while other students are able to complete this task successfully. There is a split in the students' ability to create a problem solving plan, some students are very capable of creating a plan while other students appear to be resorting to random strategies.

Table 2 indicates that the correlation between stage 2 and the other stages was significant at the 0.01 level. In contrast, stage 3 did not correlate significantly with any other stage except stage 2. Stage 2 is almost exclusively used in the traditional assessment of students' mathematical knowledge however; only stage 2 correlates significantly with stage 3. Researchers in mathematics education have commented on the use of multiple strategies to affect students' performance in mathematics, "First, there was a relationship between the number of strategies attempted by students for a particular problem and their success in solving that problem" (Pugalee2004, p.43) and "Comparing and contrasting solutions seemed to support gains in procedural knowledge because it facilitated students' exploration and use of alternate solution methods" (Rittle-Johnson & Star, 2007, p.572). The results of this data point out a limitation to this

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approach as the effect that instruction in stage 2 can have on a student's grade, which is typically measured by stage 3, is limited to the square of the correlation coefficient between stage 2 and stage 3 or approximately 7%.

If the second research question is practical for educators of mathematics, the third is at the heart of reform mathematics and those who advocate critical thinking. In stage 2, students must evaluate multiple strategies and distinguish between correct and incorrect strategies. This is frequently labeled critical thinking which Sezer (2007) describes as, "used to solve problems, choose between alternatives" (p. 350). If the role of education is to encourage students to think critically then the relevant question should be: to what extent can the other three stages affect students' ability for the abstract critical thought required in stage 2?

Table 4-A indicates that 37.5% percent of students' scores on the second stage are predicted by their performance in stages 1, 3 and 4. The data in table 4-B confirms that the use of stages 1, 3 and 4 as independent variables to predict stage 2 is a significant model at the 0.000 level. The results in table 4-C validate that stage 1 ($p=0.00$) and stage 3 ($p=0.02$) and stage 4 ($p=0.05$) are all significant factors in predicting stage 2 at the 0.05 level.

Conclusion

Adult remedial students experienced the most difficulty in the planning and monitoring stages of Polya's four step method. Their mean scores in stages 2 and 4 suggest that at most, they sometimes engage in these thought processes. Even more

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telling is that, the distributions (figure 2) of stage 2 and stage 4 indicate these students were unfamiliar with the thought processes required in planning and monitoring. The data in table 3 provides insight to this phenomenon as it shows stage 2 receives the least amount of representation in textbooks. If one extends this to include classroom presentation then the randomness suggested by the stage 2 distribution could directly and reasonably be linked to students' lack of exposure to multiple strategies.

These results indicate adult remedial students have a limited problem-solving schema and supports the characterization of these developmental college students as not being exposed to planning and thus rarely able to plan, or consider an alternate plan if their initial plan fails. In particular, while their mean score of 2.2 on stage 3 and the relatively normal distribution of that stage indicates these students are more familiar and more capable with the cognitive thought process involved in carrying out a concrete plan.

The results of the correlations in table 2 and multivariate analysis with stage 2 as dependent variable in table 4-A, 4-B, and 4-C yield some insight into how instruction in the independent variables may affect students' performance in the dependent variable. The results indicate that, if the goal of instruction is computational proficiency or the ability to recognize concrete applied strategies, then instruction in stage 2 is the only other relevant stage and this planning stage can at best predict about 7% of this goal. On the other hand if the goal is to improve students' ability for abstract planning, then instruction in the other stages can predict about 37.5% of students' demonstrated performance in this area.

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Implications for Instruction and Future Research

For mathematical educators of remedial pre-algebra there is a window of opportunity where an increase in the use of multiple strategies and metacognitive reflection will lead to an increase in students' performance. Whether with proper instruction these developmental students will effectively learn multiple strategies, engage in reflection and use these skills to increase their problem solving is an important question. Passmore (2007) notes that, "heuristic training [has] proved disappointing" (p. 48) and Schoenfeld (1987) also found, "…no clear evidence that the students had actually learned more as a result of their heuristic instruction or that they had learned any general problem-solving skills that transferred to novel situations" (p. 41).

The results of the study clear demonstrate the difficulties students have engaging in the conscious thought necessary for planning and monitoring their problem solving activities and further suggest that, remedial students may be prone to guessing what strategy to use rather than engage in the reasoning required in these stages however, to what extent this behavior is due to lack of ability or attitude is impossible to ascertain.

According to some sources, mathematics education has failed these students for it has provided them with rules and formulas but does "…not necessarily lead to mathematical thinking skills in which a pupil is challenged to solve an unfamiliar problem" (Taylor & McDonald, 2007, p.640). Other researchers argue that students' attitude toward mathematics prohibits deviating from the acceptable rules for mathematics. While more research is needed to clarify this issue, it may be the most

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critical factor in determining the success or failure of all curricula reform efforts to encourage students' conscious thought during problem-solving.

The results shown in tables 2 and 4 A-C also have important implications for mathematics instructors. If the goal or objective of an intervention is measured by students' computational proficiency or ability to understand concrete, worked out solutions of a problem (stage 3) then there is a minimal window of opportunity (only about 7%) that instruction in stage 2 can affect this outcome. If however, the goal is to increase students reasoning and abstract planning ability (stage 2) then a balanced approach of instruction in the other stages of Polya's method has a higher window of opportunity (about 37.5%) to beneficially affect students' performance.

The adult students in this study have experienced difficulties with mathematics and in particular the cognitive gap between arithmetic and algebra. Several important research questions that arise from this study, namely, would specific instruction in multiple strategies for problem solving turn the distribution of stage 2 into a more normal distribution? Would explicit instruction increase students' ability and willingness to plan strategies when problem solving? Would a balanced instruction of Polya's method show increases in students' ability to plan and recognize strategies for problem solving? To what extent does students' problem solving schema predict their future success in algebra? The answers to these questions may help close the gap that exists between students' achievement in remedial courses in mathematics.

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Appendix A

The proportional reasoning exercise is divided into four parts, one for each of Polya's stages, and the rubrics used to score each exercise are presented below. A score of zero was assigned to students who had an answer completely incorrect. The results from students who skipped a question were omitted from the overall scores for that question only.

Stage 1: Understanding the Problem

Exercise #1) Juan and Maria are making lemonade for lunch by mixing cups of sugar with glasses of water that are the same size. Maria who is on a diet uses one cup of sugar for every three glasses of water, Juan who likes sweet lemonade uses three cups of sugar for every eight glasses of water. If each glass contains exactly two cups then whose lemonade is going to be sweeter?

Exercise #1 Circle **ANY & ALL** of the following that are true

I) The amount of sugar that Juan & Maria use is necessary to answer the problem
II) The amount of water that Juan and Maria use is necessary to answer the problem
III) The fact that the glasses used are exactly two cups is necessary to answer this problem
IV) The fact that Juan likes sweet lemonade and that Maria is on a diet can be used to answer this problem
V) None of the above can be used to determine whose lemonade is sweeter.

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The following rubric was used to measure students' ability to recognize relevant information and accurately distinguished between relevant and extraneous information.

1. Understanding the Problem	1. Shows limited understanding of the problem, what information is relevant versus superficial and what you are asked to find.
	2. Shows partially developed understanding of the problem and indentifies a few specific factors that influence the approach to a problem before solving.
	3. Shows clear understanding of the problem and identifies many specific factors that influence the approach to a problem before solving
	4. Shows clear understanding of problem and identifies specific factors that influence the approach to a problem before solving. Recognizes unnecessary factors.

Stage 2: Devise a Strategy or Plan

Exercise #2) Circle **ANY & ALL** answers below that describe a strategy that can be used to solve this problem.

I) Compare the amount of sugar that Juan used to that Maria used
II) Compare the ratio of sugar to water than Juan used to the ratio of sugar to water that Maria used.
III) Find the ratio of cups of sugar to total cups of lemonade for Juan and compare this to the ratio of cups of sugar to total cups of lemonade for Maria.
IV) Use the fact Maria is on a diet and Juan loves sweet lemonade
V) Find the percent of sugar in Juan's lemonade and compare this to the percent of sugar in Maria's lemonade
VI) Not enough information given to answer this problem

The rubric was used to score the students based upon whether they could recognize multiple strategies to solve the problem and distinguish correct from incorrect strategies.

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2. Devising a Plan-Identify Strategies	1. Select a strategy without regard to a fit
	2. Identifies a viable strategy
	3. Designs or shows understanding of one or more strategies in the context of the problem
	4. Designs or shows understanding of one or more strategies, along with either articulating the decision or ability to identify incorrect strategies.

Stage 3: Carrying out the Plan

Exercise #3) Jorge and Alba need to determine which fraction is largest: $\frac{7}{9}$ or $\frac{8}{11}$

Circle **ANY & ALL** the following that correctly describe a way to determine which fraction is largest?

a) The second fraction is largest because 8 is larger than 7
b) With both fractions, divide the denominator into the numerator and then compare the size of the resulting decimals
c) Cross multiply and compare the product of the means 9x8 with product of the extremes 7x11
d) The first is largest because 9 is smaller than 11
e) Convert both fractions to equivalent fractions over the LCD = 99 and compare the resulting numerators.

Rubric used to evaluate thought process of this stage

3. Carrying out the Plan-Generate Solutions	1 Rarely recognizes the need for multiple paths to carry out the plan **minimal,** thought or reasoning in carrying out the plan can recognize or state at most potential solution method.
	2 Sometimes recognizes multiple paths to carry out the plan but reasoning or thought in carrying out the plan is not well developed will typically relate to one perhaps more potential solution.
	3 Frequently recognizes the need for multiple solutions, or multiple paths to carry out the plan reasoning or thought in carrying out the plan is well developed will typically state one or more alternate and

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	accurate (even creative) potential solution(s).
	4 Always recognizes the need for multiple paths to carry out the plan reasoning or thought in carrying out the plan is fully developed will typically state one or more alternate and accurate solution methods. Recognizes inadequate methods.

Stage 4: Looking Back

After answering the question of whose lemonade is sweeter, what can be accurately said about the process involved in answering this problem?

Exercise #4) Circle **ANY & ALL** of the following which are true:

a) The sweetness of lemonade depends only upon how much sugar was used.
b) The Sweetness of lemonade depends upon the rate of sugar to water and this can be expressed as a fraction.
c) Comparing the sweetness of two separate lemonade mixtures involves proportional reasoning between the rate of sugar to water in each mixture.
d) The sweetness of lemonade depends upon the rate of sugar to water and this rate can be expressed as a decimal.
e) To accurately compare the sweetness of separate batches of lemonade they must both use the same amount of water.
f) All of the above are true.

The following rubric was used to evaluate students' reflection on what was involved in solving the plan

4. Looking Back	1. Does not analyze or synthesize results, shows limited understanding of what was involved in solving the problem.
	2. Sometime analyzes or synthesizes results, shows some understanding of what was involved in solving the problem.
	3. Frequently analyzes or synthesizes results from more than one perspective, shows good understanding of what was involved in solving the problem.

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	4 Always analyzes or synthesizes results from a wide range of perspectives, shows good understanding of what was and able to discern what was not required for solving the problem