MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

Developing Pre-service Elementary School Teachers' Conceptual Understanding of Addition and Subtraction: An Exercise in Base Four

Dr. Lidia Gonzalez

York College of the City University of New York

U.S.A.

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

Abstract

This article describes a lesson taught by the author to a class of pre-service elementary school teachers. One of the author's goals in teaching the lesson was to deepen the pre-service teachers' conceptual understandings of the operations of addition and subtraction on whole numbers. In the lesson students were introduced to base ten blocks. They were later asked to use the same methods to add and subtract in base four and were given manipulatives to aid them as they did so. The author, after describing the lesson, reflects upon students' reactions to the lesson, its effectiveness, and what characteristics of the lesson may have contributed to it being valued by both the students and the author herself. Implications for teacher educators are also considered.

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

Introduction

This article describes a lesson that I taught as part of an undergraduate mathematics course specifically for pre-service elementary school teachers and how these pre-service teachers responded to it. Using this lesson and the experiences of both the students in the class and myself as the instructor, I consider the ways in which this class' experiences can inform the work of teacher educators, especially those working with pre-service elementary school teachers.

My students, all but one of who are pre-service elementary school teachers (one student is taking the course as an elective) are not shy about their dislike for and difficulty with mathematics. Although in many cases they are extremely capable of carrying out standard algorithms for solving various problems, they do not always have an understanding of the concepts that drive these algorithms. Thus, while all can add a three digit number to a three digit number by following the steps they have used time and again, they do not seem to understand this addition in ways that transcend the steps involved. Simply put, while they have a strong procedural knowledge about much of the material, most lack a conceptual understanding of it. This is a common issue with pre-service teachers as well as the population at large. Lloyd (2006) argues that, "many preservice teachers possess weak knowledge and narrow views of mathematics and mathematics pedagogy" and furthermore, because "such conceptions deeply affect the learning-to-teach process teacher educators are faced with the task of creating opportunities for preservice teachers to develop useful, dynamic conceptions of mathematics and pedagogy" (p. 12).

Though there is debate about what exactly a teacher needs to know in order to teach mathematics effectively, there is evidence that, "how well teachers know mathematics is central to their capacity to use instructional materials wisely, to assess students' progress, and to make sound judgments about presentation, emphasis and sequencing" (Ball, Hill & Bass, 2005). It should be noted that a deep conceptual knowledge of the mathematics is a necessary, though not sufficient, condition for effective teaching. Furthermore, given the research on mathematical knowledge for teaching, it is clear that, "knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the [mathematical] algorithm reliably" (Ball, Hill & Bass, 2005, p. 21). In fact, the teaching of elementary school mathematics demands "knowledge of mathematics that is rooted in conceptual understanding and in the modes of inquiry of the discipline" (Crespo & Featherstone, 2006, p. 109). Fostering a deep conceptual understanding of the mathematics is helpful for when these pre-service teachers later enter the teaching profession as it allows them to exploit rich opportunities for student learning and take advantage of teachable moments in ways that those with more limited mathematical knowledge cannot (Vatuk & Meagher, 2009). Thus, preparing my students, most of which are pre-service teachers, in ways that develop their conceptual understandings of the mathematics involved is a goal of mine for the course.

Context of the Course and Lesson

The undergraduate course in which the lesson took place is a required course for early childhood/elementary education majors. As students are formally accepted to the

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

education program in their third year, it is usually taken in one's junior or senior year. The focus of the course is on mathematical content and in it I attempt to review and expand upon students' understanding of mathematics in a wide variety of areas including number sense, operations on integers and fractions, probability, statistics, logic, ratio and percents. The goal is for students to leave with an understanding and appreciation of a wide range of mathematical topics that they may be expected to teach when they enter the teaching profession. That is, through this course I attempt to support pre-service elementary school teachers so as to develop a deep, conceptual understanding of the mathematics they will one day teach.

The course is one of two in the students' major with a focus on mathematics. After completing this course, students are expected to complete a math methods course where they work on how to successfully teach those mathematical concepts learned in the prior course. Thus the focus of the second course is pedagogy while the focus of the first is mathematics content though these do over lap. As the students in the content course are, for the most part, studying to be teachers, in this course I aim to also model successful pedagogical techniques and introduce students to manipulatives and other educational aids.

This article focuses on a specific lesson where the learning objective was that students develop a conceptual understanding of the operations of addition and subtraction. My students are familiar with our base ten system as well as the standard procedures for adding and subtracting, and have used both successfully throughout their lives. The students were all familiar with the traditional US algorithm for adding and

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

subtracting multi-digit whole numbers, but one objective of this course was to transcend that algorithm. Specifically we were looking at addition as grouping and regrouping, and looking at the concept of borrowing in subtraction again by relying on the concept of grouping. We approached these concepts first through the use of base ten blocks. For those unfamiliar with base ten blocks and their use, a brief introduction follows.

Base Ten Blocks

Base ten blocks are manipulatives consisting of plastic three-dimensional solids each of which represents a power of ten. There are four pieces called respectively unit, rod (or long), flat, and cube (which in class we called block). The value of each piece is given by its volume. The smallest piece is called a unit. It is a small cube each face of which has sides of length 1 unit. The volume of the piece as well as its value in our number system is 1 (that is, 10^0). If we arrange ten units end to end, we create the equivalent of a rod. Thus, a rod is a rectangular prism with volume 10 (that is, 10^1). The rods are scored so it is clear to see that the piece was created by putting 10 units together. If one puts 10 rods side by side the equivalent of a flat is created. A flat is a rectangular prism with volume 100 (that is, 10^2). Again the piece is scored so as to make clear that it was created by putting 10 rods or 100 units together). The final piece is called a cube, each face of which has a side length of 10. It can be created by stacking 10 flats one atop the other. Its volume and value is 1000 (that is, 10^3).

A number of no more than 4 digits can be represented using these blocks. Thus, 1234 is represented by 1 cube, 2 flats, 3 rods and 4 units. Numerically, we have

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

$1234 = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0$. Using the blocks we can model addition and subtraction of whole numbers of no more than four digits. That is, those less than or equal to 9,999. Let us now consider an example of addition in base ten modeled through the use of the blocks.

Addition with base ten blocks

Let us find the sum of 123 and 98. We begin by representing 123 using the blocks. One hundred twenty three is represented using 1 flat, 2 rods and 3 units. The number 98 is represented by 9 rods and 8 units. Putting all these pieces together gives us the sum of the two numbers. Thus addition is the grouping together of the representations of these two numbers. This yields, 1 flat, 11 rods and 11 units. However, using the blocks we can exchange 10 of the 11 units for a rod. This gives us 1 flat, 12 rods and 1 unit. We can then exchange 10 of the 12 rods for a flat. This leaves us with 2 flats, 2 rods and 1 unit making our sum 221. As we are working in base ten, each digit used is between 0 and 9 inclusive. The regrouping ensures that the number of units, rods, flats and blocks can be represented using one of these digits.

Additionally, with the blocks one can see the algorithmic step of carrying the 1 in a traditional addition problem through the ideas of grouping and exchanging. A young student may not understand initially why we carry the 1 when adding but may be able to understand more easily in terms of exchanging 10 units for a rod or tens rod for a flat, for example, which is one reason these blocks are used to model addition when teaching young children. Addition understood in this manner becomes a matter of putting items

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

together, grouping them ten at a time, and using equivalent representations through exchanges.

Subtraction with base ten blocks

Let us now look at subtraction using base ten blocks. Take the example $123 - 98$. We begin, as we did with the addition by representing 123 using 1 flat, 2 rods and 3 units. We are being asked to subtract 9 rods and 8 units. The subtraction can be modeled by literally removing the pieces from the table upon which they are placed. However, at the start we have 3 units and need to take away 8. The representation currently on the table makes it impossible for us to do so. We can, however, exchange a rod for 10 units. Doing so gives us 1 flat, 1 rod and 13 units. After removing 8 units we are left with 1 flat, 1 rod and 5 units. We still need to remove 9 rods but only have 1 at present. Again we need to exchange. In this case we exchange a flat for 10 rods and now have 0 flats, 11 rods and 5 units. Taking 9 rods away leaves us with the answer to our problem: 2 rods and 5 units. That is, 25.

The Lesson

With the goal of developing a conceptual understanding of the operations of addition and subtraction, I began to consider using base ten blocks to model problems involving these operations. The blocks can be used to support young learners as they move from the concrete process of using them to the more theoretical process of adding and subtracting by using the standard algorithms. Teachers of elementary school students need to understand the use of these blocks in order that they may effectively use them with their students. My students as prospective elementary school students would, in my

TEACHING–RESEARCH
NYC MODEL
MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

opinion, benefit from familiarity with these blocks. Despite these reasons, I was hesitant to use the blocks with my own students because I was concerned that having them use the blocks to add numbers they can easily work with in other ways might not be effective. I feared they would go through the motions with the blocks but mostly rely on their comfort with base ten arithmetic and the standard algorithms and as a result would not be pushed to deeper understandings. I worried that, at most, they would walk away with an interesting way to teach a student how to add or subtract which though important, was not the goal of the lesson. I feared that the students' conceptual understandings of the operations of addition and subtraction of whole numbers may not develop beyond what they already held.

Driven, in part by the above and in part by the fact that I had yet to discuss number systems with the class, I decided to develop a lesson that would involve their working not in base ten, but in base four, with the hopes that through this work, they would begin to understand addition and subtraction in new ways.

The lesson began with my introducing base ten blocks to the class in a similar manner to that described earlier in this paper. We named each of the pieces, talked about each piece's volume and how this equaled the number that the block represented. We put units together to form rods and rods together to form flats. We discussed why there was no individual piece that represented 10,000 and what the limitations of these blocks were in terms of representing whole numbers. We then used the blocks to represent numbers and related the representation with blocks to a number's expanded notation. Finally, we turned our attention to adding and subtracting using these blocks.

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

I demonstrated to the class how the operations could be modeled with the base ten blocks and how specifically carrying over and borrowing, could be explained through grouping, regrouping and exchanging pieces. We looked at a few specific examples and I modeled each with the blocks. After this I asked the students to solve addition and subtraction problems of their own using blocks. I wrote a handful of problems on the board and asked them to work in groups to solve them. However, just before they got up to join their group members, I told them that they would be working in base four, not base ten. I explained, that this meant they would be creating groups of 4. Thus, four units would form a rod and four rods a flat and four flats a cube.

The task

Students were given plastic cubes that they could fit together. We treated each plastic cube as one unit. Students then fit the pieces together themselves in order to form rods, flats and cubes. In this way they were creating their own set of base four blocks from the given cubes and prior to modeling the different problems that were posed to them.

The students, working in groups of three to four, began modeling using these cubes, a list of addition and subtraction problems that I provided. The first problem they were to work out was $22+3$. Although most every group successfully modeled 22 by using 2 rods (each with 4 units in it) and 2 units and 3 by using 3 units, every single group indicated that the answer was 25. Although each student created the appropriate representation of the given addends, they still relied on their familiarity and comfort with base ten notation as they worked towards solving the problem. That is, they did not

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

completely work in base four. A rod in their mind was worth 10, even if they constructed it by using only four units. Group by group I went around questioning their answer of 25. I pleaded with them to use the cubes and the model that they had built. I asked them not to say "22" but to instead say "2 rods and 2 units" hoping this would lead them to think about what exactly a rod meant and not fall back on their familiarity with the base ten system. I asked them how many cubes it takes to make a rod and in time all said 4. I then asked them what they could do with the five cubes they had on their desk as they began the problem of adding 22 and 3 in base four. Although it was a challenge to most, they slowly began speaking of the problem in terms of units and rods and regrouping the pieces they had using groups of four rather than ten.

Although there was much frustration on the part of students with respect to the activity, they continued to work at it and question why their solutions were not initially correct. Eventually one student very loudly exclaimed, "Wait – I think I get it." He had a large smile plastered across his face as he explained the problem to his group members. When he had finished doing so he and the other members of the class were called over by other students so that they could share what they knew about the problems. There was a nice buzz in the room as students talked about the problems and the representations with one another, and as the blocks were moved around, built up and then separated.

In the midst of this I glanced at the clock. The activity had taken longer than I had planned and it was now minutes before the class was to end. I made a general announcement that students keep working on this at home and that if anyone wanted to stay and keep working I would wait until the last person left. One student got up and

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

walked out of class. The remaining students kept working. I circled the room answering and asking questions as did some students. Others went through the problems with their classmates. Every now and again we would hear another, "Oh, I got it" and see another big smile. At one point I looked at the board where two students stood working out some of the problems.

Eventually the two came to me and explained that they figured out how to do the problems without the blocks. That is, they developed an algorithm for adding and subtracting in base 4 similar to the one that exists in base 10. They explained how to borrow and carry over in base 4 indicating that one does this at 4 instead of 10. Their realizations are akin to those we would expect to see in elementary school students as they learn to add whole numbers. First there is a reliance on the blocks and manipulatives which later leads to an understanding of the problems and the ability to rely on an algorithm that no longer involves manipulatives.

In time, everyone walked out knowing how to do the problems. Many were able to understand and explain addition and subtraction using grouping, regrouping and exchanging. I found this to be the value of the activity and, in some way, a partial attainment of the goals the lesson set out to accomplish. Students left feeling accomplished and I left feeling that their conceptual understanding had been enhanced. Working in a base unfamiliar to them, pushed their understanding of the operations or addition and subtraction further. As the students' frustrations about why 22+3 was not 25 (in base four, that is), their curiosity grew and they persisted in their work with the manipulatives that they were given. Understanding how to model addition and

TEACHING–RESEARCH NYC MODEL
MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

subtraction in base 4 using cubes led to a deeper understanding of what it means to add and subtract more generally.

Reflections

Unlike what has happened in other lessons, I was not asked a single time why we needed to do this, whether one would ever have to explain base four addition and subtraction to elementary school students and why knowing the standard algorithm for addition and subtraction was not enough. In talking to my students about the activity days later, I learned why they felt it valuable and why they struggled with it until they could complete it. Several explained that the problem was challenging but having the manipulatives and being able to work with others made them feel they could accomplish it. That is, they felt supported in their work and this allowed them to persevere.

Some commented that their frustration might be akin to that of their elementary school students and valued this experience, in part, because of this. They began to consider how a young child might struggle with these operations and specifically with borrowing and carrying over which they had not fully considered previously. Struggling in base four, as some said, led them to value the struggles of young learners. Additionally, the lesson also provided them with ways to explain these concepts to students as they would be expected to do in the future and modeled pedagogy that one would hope to see them utilize in their classes once they enter the teaching profession.

Implications

The activity described here and the response of the students to such an activity has implications for teacher educators and courses geared to preparing pre-service teachers.

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

With respect to this, there are two points that may serve to inform teacher preparation programs and those who teach within them.

First, these pre-service teachers valued a challenging activity in part because of the support that they were given to complete the task. The fact that they worked with their classmates, had consistent check-in from me and had access to manipulatives on which they could rely, gave them a sense that they could achieve what was required. Often we hear of the importance of having high expectations for our students – that students rise to the level of expectation placed on them. Yet if high expectations are not coupled with adequate supports, students are left frustrated. In those cases, students with negative perceptions of mathematics and more specifically negative beliefs with respect to their ability to do mathematics will have those beliefs reinforced. However when adequate supports are in place, challenging material is met with interest, curiosity and persistence. Thus when we work with pre-service teachers we must keep in mind that high expectations should be coupled with support if students are to rise to the challenges, expand their understanding and leave the classroom feeling accomplished. We often remind pre-service teachers of this so they may keep it in mind as they teach and we must follow our own advice. Additionally Lloyd argues that "over time, most preservice teachers come to appreciate the place of investigation and discussion in the development of mathematical understanding" and that discomfort with these activities lessens with time (2006, p. 19).

A second implication has to do with the level of mathematics that the lesson contained. In some cases, teacher educators may wish to work with their classes on

TEACHING–RESEARCH NYC MODEL
MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

problems and activities that the students can in turn use in their classes. Though this is appropriate and does help pre-service teachers build up a repertoire of problems and activities, it is also important to challenge students with the content. My students enjoyed being challenged. Although initially frustrated, they persisted through the activity. The activity described here could have easily been done in base ten, using blocks and modeling a way in which elementary school teachers can support their students' understandings of addition and subtraction. Having them work in base 4 made the problem more challenging and better suited for the development of conceptual understanding. They were not able to rely on their procedural fluency with base ten. However, working in base four did not take away from their understanding of how to use base ten blocks to teach addition and subtraction in base 10. In fact, I would argue that it strengthened their ability to do so because they could no longer rely simply on their procedural fluency with addition and subtraction. Thus another implication for teacher educators is that when working with pre-service teachers we should not restrict ourselves to those problems that they can use with their young students, although there is a place for these as part of our courses. We should try to infuse our lessons with activities that mirror those they can do with their younger students but which challenge them mathematically. Doing so may put them in a position where their understanding of the material is deepened. It might also, as was the case with my own students, put pre-service teachers in a position where they are better able to understand their own students' struggles and successes.

MATHEMATICS TEACHING-RESEARCH JOURNAL ONLINE
VOL 3, N3
August 2009

References

Ball, D. L., Hill, H. C. & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, Fall 2005*, 14-17, 21-22, 43-46.

Crespo, S. & Featherstone, H. (2006). Teacher learning in mathematics teacher groups: One math problem at a time. In K. Lynch-Davis & R. L. Rider (Eds.), *The work of mathematics teacher educators: Continuing the conversation*, monograph series, 3 (pp. 97-116). San Diego, CA: Association of mathematics teacher educators.

Lloyd, G. M. (2006). Using K-12 curricular materials in teacher education: Rationale, strategies, and preservice teachers' experiences. In K. Lynch-Davis & R. L. Rider (Eds.), *The work of mathematics teacher educators: Continuing the conversation*, monograph series, 3 (pp. 11-28). San Diego, CA: Association of mathematics teacher educators.

Vatuk, S. & Meagher, M. (2009). *Pedagogical choices and mathematical knowledge for teaching of NYC teaching fellows.* Manuscript submitted for publication.