Mathematics Teaching-Research Journal On-Line

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

Using the Concept of Upper and Lower Bounds to Find Square Roots

Nkechi Agwu

Introduction

This paper provides a short lesson that can be used to reinforce students' understanding of square roots. Through this lesson, students will review the definition of square roots and the chronological history of the square root symbol. They will use two ancient Egyptian methods involving upper and lower bounds to approximate square roots and possibly facilitate their understanding of the concept of a limit. Students' will solve quadratic equations by applying these two methods. This lesson requires students to have some pre-requisite knowledge of real and imaginary numbers, perfect squares, and upper and lower bounds, or the instructor to introduce the afore-mentioned mathematical ideas. The concepts of upper and lower bounds and limit are fundamental ideas in mathematical analysis, so early reinforcement of these concepts are crucial for student success in calculus.

Definition of Square Root

Let x and N be real numbers, with $N \geq 0$.

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

Then x is the square root of N, if and only if the square of x is N.

Symbolically $x = \pm \sqrt{N} \rightarrow X^2 = N$.

The principal square root of N is $\sqrt{N}$.

If $N < 0$, then $\sqrt{N}$ is not a real number. It is an imaginary number.

For example, $\sqrt{-16} = \sqrt{[(16)(-1)]} = (\sqrt{16})(\sqrt{-1}) = \pm 4i$ or $(\pm 4i)^2 = -16$

Examples:

(1) Find the square root of 9.

Answer: $\sqrt{9} = \pm 3$ since $(\pm 3)^2 = 9$

(2) Find the principal square root of 9.

Answer: The principal $\sqrt{9} = 3$

(3) Find the square root of –9.

$\sqrt{-9} = \pm 3i$ since $(\pm 3i)^2 = -9$

Chronological history of the square root symbol

A discussion of the chronological history of the square root symbol provides students' with the perspective of mathematics as an evolving discipline and with an appreciation of the contributions of different people and cultures to the development of mathematical concepts. The chronological history of the square root symbol provides a

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

global perspective to the concept of square root. The history indicates that the square root symbol whose origin can be traced back to the ancient Egyptian civilization (3000B.C. – 400 A.D.) in Africa showed up later in the 15th century mathematical traditions of Europe in France and Germany, undergoing various evolutionary forms and finally transitioning to the symbol that we use in today's modern society.

(1) In ancient Egypt, A'homose in the *Rhind Papyrus* used the symbol " ⌈" to denote square root.

(2) In France, Nicolas Chuquet (d.1487) used "$R^2 6$" for the square root of 6.

(3) In Germany, Michael Stifel (ca.1487-1567) developed the symbol "√·" and later refined it to "√ ".

(4) In France, François Viéte (1540-1603) used "l" to denote square root, but his follower Franz van Schooten adopted the radical sign "√ " in 1646.

(5) In the 16th century, Descarte used the symbol "√ " to indicate square root. This is the symbol we continue to use today to indicate square root.

Finding Square Roots Using Ancient Egyptian Methods of Upper and Lower Bounds

Mathematics Teaching-Research Journal On-Line

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

In ancient times, our ancestors did not have calculators. So, they used approximate methods to calculate the value of the square root of a number. Given below are a few of these methods.

Method 1 - Upper and Lower Bounds

Step 1

Determine the number of the digits in the integer part of the square root.

This will involve upper and lower bounds for the integer part that are powers of 10.

For example, for 10^n where n is a whole number, there is a number x for which $1 \leq x < 100$.

Observe that $1 \leq \sqrt{x} < 10$.

The square root of x will only have one digit left to the decimal point.

Step 2

Select the number of decimal places of interest.

Then write the square root as A.BCD…, where A is the integer part and .BCD… is the decimal part.

Step 3 and subsequent steps

Proceeding step by step using upper and lower bounds as explained below.

Mathematics Teaching-Research Journal On-Line

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

Determine the square root to the given number of decimal places.

Example 1: Find the solution of the quadratic equation $X^2 = 1000$.

Solution: To answer the question, we need to find the square root of 1000 to the second decimal place.

Step 1

Since $10^2 < 1000 < 100^2$ then, $10 < \sqrt{1000} < 100$.

The square root of 1000 has 2 digits number left to the decimal point, so it has the form as AB.CD.

Step 2

Find out the first digit number A.

$10^2 = 100$, $20^2 = 400$, $30^2 = 900$, $40^2 = 1600$,…

900 is too small for 1000, and 1600 is too big for 1000, so we can easily tell the square root of 1000 lies between 30 and 40, and the first digit in its integer is 3, written as 3B.CD…

Step 3

Find out the second digit B.

$30^2 = 900$, $31^2 = 961$, $32^2 = 1024$,…

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

961 is too small for 1000, and 1024 is too big for 1000, so we can tell the square root of 1000 is between 31 and 32, and the second digit must be 1, written as 31.CD…

Step 4

Find out the first decimal number C.

$31.1^2 = 967.21$, $31.2^2 = 973.44$, $31.3^2 = 979.69$, $31.4^2 = 985.96$, $31.5^2 = 992.25$, $31.6^2 = 998.56$, $31.7^2 = 1004.89$,…

You will see that $998.56<1000<1004.89$, so the square root of 1000 lies between 31.6 and 31.7. The first decimal number is 6, the value is written as 31.6D…

Step 5

Find out the second decimal number D.

$31.61^2 = 999.1921$, $31.62^2 = 999.8244$, $31.63^2 = 1000.4569$,…

The square root of 1000 lies between 31.62 and 31.63, the second decimal number D is 2, written as 31.62…

Now, we've found out the square root of 1000 to the second decimal places, so the solution is x = 31.62… to the 2nd decimal place. This is fairly accurate, but it is terribly slow and inconvenience, since it requires a huge amount of calculation. Also errors may occur due to the miscalculation.

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

Exercises:

Find an approximate solution for the given quadratic equations to the given decimal places. Use the square root method involving upper and lower bounds.

(1) Solve the quadratic equation $x^2 - 1570 = 0$, remain accuracy to 2nd decimal place.

(2) Solve x for $x^2 - 442 = 0$, remain accuracy to the 1st decimal place.

(3) Solve x for $x^2 - 8 = 0$, remain accuracy to the 3rd decimal place.

(4) Solve x for $x^2 - 56.32 = 0$, remain accuracy to the 2nd decimal place.

(5) Solve x for $x^2 - 1.72 = 0$, remain accuracy to the 3rd decimal place.

Answers: (1) 39.62 (2) 21.0 (3) 2.828 (4) 7.5 (5) 1.311

Method 2 - Trial and Error Method for a Perfect Square

When you are asked to find the square root of a number that has a perfect square root, you can use the method below.

Problem:

Solve x for $x^2 - 1156 = 0$.

$X^2 = 1156$

$X = \pm\sqrt{1156}$.

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

STEP I:

Determine two perfect squares, a close lower bound of 1156 and a close upper bound for 1156. Since $30^2 < 1156 < 40^2$, then $30 < \sqrt{1156} < 40$.

STEP II:

Look at the last digit of the number that you are working with and try to find out several possible choices for the principal square root.

The number 1156 ends in six.

So, we determine the digits from 0 to 9 whose square ends in six.

$0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$

This gives two possible choices for the last digit of the actual value of the principal square root of 1156.

We may guess the answer be 34 or 36, because they will end in 6. ($4^2 = 16$, $6^2 = 36$.)

Then we calculate the $34^2 = 1156$, $36^2 = 1296 > 1156$.

So, the square root of 1156 is ± 34.

Exercises:

Find an approximate solution for the given quadratic equations by the trial and error method for a perfect square.

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

1) Solve x for $x^2 = 3844$.

2) Solve x for $x^2 = 7569$.

3) Solve x for $x^2 = 8281$.

4) Solve x for $x^2 = 2116$.

5) Solve x for $x^2 = 13225$.

Answers:

(1) 62 (2) 87 (3) 91 (4) 46 (5) 115

Inquiry-based Activities:

1. Ask students to reflect on the strengths and limitations of these methods.
2. Ask students to research other historical methods of finding square roots by approximation techniques.
3. Ask students to provide a comparative analysis of two historical methods of finding square roots using approximation techniques.

Bibliography

(1) Agwu, N., *History of Solving Quadratic and Polynomial Equations*, unpublished paper, 2000.

A peer-reviewed scholarly journal
Editors: Bronislaw Czarnocha (Hostos Community College)
Vrunda Prabhu (Bronx Community College)
Anne Rothstein (Lehman College)
City University of New York
Volume 2 Date March 5 2007

(2) Agwu, N., Frey, P., Greer, T. and Taylor, G., *Polynomials Module*, Washington D.C.: Mathematical Association of America Institute in the History of Mathematics and its Uses in Teaching (MAA-IHMT) Historical Modules Project, 2001.