Modeling, Schemas and Creativity in the Learning of Differential Equations

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Introduction

Differential equations have many applications to real problems. Even though the way courses on this discipline have changed and technology and applications have been introduced regularly, applications are generally introduced by describing the problems and showing adequate mathematical models using differential equations to solve them. Students are seldom given the opportunity to struggle with real problems and find possible ways to model them by themselves.

There have been some studies that show how modeling can be used in a differential equation course (Kwon et al., 2005; Chaachua & Saglam, 2006; Rowland, 2006; Trigueros, 2014). Results of these studies show that using models in the classroom helps students develop methods and concepts they can use as tools to develop their knowledge.

The goal of this paper is to explore how creativity plays a role in students' development of knowledge. APOS theory and Koestler ideas about creativity are linked to analyze an episode occurring within a class where models were used to introduce differential equations. The work of a team of students is used as a case study to explore those instances where new ways to look at the situation appear in terms of schemas and creativity.

The research question guiding this research: What role does creativity play in students learning when students are given the opportunity to work with real and open problems?

Theoretical framework

APOS Theory is a well know theory of Mathematics Education which started with the work of Dubinsky. Starting from Piaget's genetic epistemology, definitions are given to those constructs that explain the construction of mathematical knowledge. Although this theory can be considered as cognitive, it includes as one of its fundamental objectives to stimulate advanced mathematical concepts learning by developing pedagogical activities in the classroom on theoretical models validated from research results obtained from experiences based on APOS theory. This theoretical approach has been applied, since its creation, to the study of multiple mathematical concepts, and it has been shown that the pedagogy strategies based on the theory are successful in terms of learning (Arnon et al; 2014).

The main constructs of the theory are the structures needed in the construction of mathematical concepts: Actions, Processes, Objects, and Schemas, and with the mechanism involved in the construction of those structures: reflective abstraction. Together, they are used to describe the construction of mathematical knowledge. These constructs are defined as follows (Arnon et al; 2014).

Actions are transformations of previously constructed cognitive objects that students perceive as external or as a series of instructions required to perform each operation in an algorithm. They can be very limited, but they are crucial in the construction of any mathematical concept. When actions are repeated, students can reflect upon them and imagine or carry out them without following external stimuli; when this happens it is said that actions have been *interiorized* into a *Process*. Students who show this type of construction have more control over the transformations they are required to apply. Different processes can be *coordinated* to construct new processes and can *reverse* a process. When students need to apply actions on a process and they are able to conceive it as a whole, they can *encapsulate* the process into a cognitive *Object*. Students who have constructed a cognitive object can apply new actions on to apply it to other contexts, to find its properties or to relate it with other already constructed objects. Students can de-encapsulate an object to use the process involved in its construction.

A *Schema* for a mathematical concept is a collection of actions, processes, objects, and other schemas that can be used in the solution of classes of mathematical problems. These structures are constructed by means of relations, coordinations and transformations among the different structures mentioned above. Schemas are dynamic structures that are continuously changing. The development of the schema is described in APOS theory by means of Piaget's triad stages (Piaget & Garcia, 1989). The *Intra-stage* is manifested through the existence of isolated or weakly related structures. In the *Inter-stage* transformations are developed among the structures composing the schema; structures are grouped and can even be identified by the same name. The *Trans- stage* is characterized by the construction of synthesis among the structures in the schema, or a unifying principle; there is awareness of the relations and transformations constructed before and it can be said that the schema is a coherent structure in the sense that the students can decide if a particular schema can be or cannot be used in a specific situation. Schemas can become mental objects, the mechanism involved in this transformation is *thematization*. The construction, and development of different schemas and the construction of relations between them make the development of mathematical knowledge possible.

A genetic decomposition in APOS theory is a theoretical model whose role is to predict how a particular concept or set of concepts is constructed. It is a description of the structures and mechanisms that researchers consider are needed in the construction of the concept or concepts of interest. It can also include those relations among actions, processes, objects, and schemas needed in the construction of a schema. Genetic decomposition of a schema describes the structure included in it together with the type of relations, transformations and unifying principle needed in its development.

Genetic decompositions have to be experimentally tested by research studies. They can then be validated or refined through making some changes in accordance with the results obtained from students' responses in research studies. As the genetic decomposition is a model, it is not necessarily unique. Several genetic decompositions for the same concept or concepts can co-exist. What is important is that they accurately predict students' constructions when they are learning a mathematical concept.

When APOS theory is described, it may seem that the construction of knowledge follows a linear progression of structures. This is not the case since there is a dialectic progression in the constructions where there can be partial developments, transitions and regressions from one structure to the other. What is clear in APOS theory is that the way a student works with mathematical tasks can be related to the construction of specific structures. (Czarnocha, Dubinsky, Prabhu, & Vidakovic, 1999).

The didactic methodology of APOS Theory is called the ACE teaching cycle. It starts with students working with Activities where they can do actions and reflect upon them to construct the other structures suggested by the theory. Work on activities is done by students doing collaborative work in teams. These activities are followed by whole group discussion with the teacher in Class in order to give more opportunities to students to make the needed constructions and to formalize the concepts. Exercises are also given to students as homework so they can progress in the construction of the concept or concepts. This cycle can be repeated as many times as needed to give opportunities to students to construct all the structures included in the genetic decomposition. (Arnon, et al; 2014).

APOS theory has been used in many investigations regarding constructions that start by doing actions on previously constructed objects. However, there are few research studies on the possibility to construct mathematical objects by doing actions on other mental objects that are not directly related to mathematics but that can play a role in the construction of mathematical objects and schemas (for example, Trigueros, 2008; Trigueros & Possani, 2013). This kind of construction can take place when modeling is used in the introduction of mathematical concepts. (2014) has proposed a possible way this can happen in a description of the modeling process using APOS Theory: When students face a problem where modeling is necessary, they use their mathematical schemas and coordinate them with schemas they have constructed in other knowledge domain that can be useful in the solution of the problem they face. Students take from these schemas those structures they need to select variables of the problem and to implicitly or explicitly formulate their first hypothesis about the behavior of the studied phenomenon, and their possible simplification and mathematical expression. Starting from those hypotheses, it is possible to perform actions on the mathematical and extra-mathematical variables to establish relations between them. These actions are interiorized into processes that make it possible to manipulate and transform the original relations. Processes from extra- mathematical schemas are coordinated with processes which are components of the mathematical schemas. The result of that coordination is an emergent schema describing a mathematical model that can be encapsulated into an object. Actions and processes are then performed on the model. They make it possible to analyze the model, determine its properties and ask new questions that may lead to refine or modify the model. This cycle can be repeated until students find a model they consider appropriate to answer the questions they had posed to understand the original problem situation.

While working with the model, it may be necessary to construct new mathematical knowledge. In this case, the construction of those concepts can be described in terms of APOS theory as has been described before. During these cycles, the original schema develops through the interaction of the extra-mathematical and the mathematical schema. Its development can be described by the triad progression.

Interesting theoretical questions arise in this context. Is it possible to relate this description to Koestler's theory of creativity (1964)? Is it possible to apply this theory to the modeling activity in order to better understand creativity and foster creative ways of thinking through the use of models and projects in the classroom? Is it possible to develop modeling activities that stimulate creativity and schema development?

In Koestler's theory, creativity is related to the possibility of an individual to observe an analogy among different unrelated phenomena or schema, which he calls matrices for experience. He calls this possibility "bisociation" by saying "I have coined the term bisociation in order to make a distinction between the routine skills of thinking on a single plane as it were, and the creative act, which… always operates on more than one plane (Koestler, 1964, p- 36) or "the perceiving of a situation or idea…in two self-consistent but habitually incompatible frames of reference". In Koestler description, bisociation is manifested through a "moment of understanding" characterized by a pattern of activity governed by a set of rules which he calls a "matrix of thought". Although his description of those matrices is flexible, the whole description brings to mind APOS description of schema development and, interestingly, the description of modeling as the construction of a new schema though the coordination of intra and extra-mathematical schema. At each stage of the schema development, according to APOS theory, individuals reflect on their work and become aware of transformations that foster the transition of a schema from the recognition of relations to the recognition of transformations needed in the leap from an Intra-stage to an Inter-stage and from the Inter- stage to a Trans- stage of development. Another important moment of insight occurs when a particular schema is recognized as a whole and is thematized. These transitions are marked by the individual's possibility to reflect on his or her own thinking process and, according to Piaget, the individual is aware that something different has emerged. In other word new knowledge is constructed and accommodation of his or her schema has occurred. This whole process is what in Piaget theory is known as constructive generalization (Piaget & Garcia, 1989). As Piaget, Koestler considers such consciousness as essential to originality and creativity, and changes in matrices of thought are changed by a "moment of understanding" fostered by bisociation of different matrices of experience (Baker, 2014). There is indeed a parallelism between Koestler description of creativity and APOS theory of schema development.

Problem solving and modeling are contexts where creativity can play an important role. According to Koestler, working with problems may need the identification of features of problems that remind the solver of other similar situations, although rules applied to the known situation are not applicable to the new one. This description is in line with the description of modeling activity using APOS theory described above: when the solver brings different schema to bear with the complexities involved in the solution of a problem, he or she may tackle the new situation by putting into play different previously constructed schemas, which can be both mathematical or one mathematical and other extra-mathematical related to other field of knowledge. To work with both schemas, the solver needs to consider their components and construct relations among them. The schemas interact (Baker et, al. 2000; Cooley et, al. 2007; Trigueros & Escandón, 2008) and it is possible that a new schema emerges from this interaction, that is, a constructive generalization. Using Koestler ideas, a bisociation occurs that manifests itself through an act of understanding, from which a new way to work with the situation and new understanding emerge.

It seems possible, then that creativity, as described by Koestler work plays a role when students work with real or open problems, and that moments of understanding resulting from bisociation can be linked to schema development. In what follows an episode of students working with a real problem in the classroom is analyzed using these ideas. The focus of the analysis will be those moments where the emergence of knowledge seems to be occurring.

In this paper we apply Koestler ideas on creativity, together with the notion of schema in APOS to analyze the emergence of new concepts and ideas in a group of students in the context of modeling in the classroom

Methodology

Research was conducted within a Differential Equations course for students in an Economics program at university level, at a small private university in Mexico. The modeling project the students worked on was conceived to be developed, both, during class time and outside the classroom during a period of about a month.

The modeling activity proposed was to develop a mathematical model to predict the price of a good in a market with price expectations. Students could select the specific good they wanted to study. As part of the project, students had to turn in a report to the manager of a company where the chosen good was produced. This report had to include the proposed model with a clear explanation of all the factors considered in its development; data collection; use of data to test the models' predictions, and a conclusion about the quality of those predictions and a discussion of any change students considered important to make to the model in case they considered it was not good enough.

Students worked collaboratively in small groups of three students, and information of all the documents produced during the modeling cycles, as well as from observation of groups and whole class discussion was analyzed by the researcher and discussed with two teachers for triangulation. Interviews were conducted with groups of students to get a better insight on what they were doing and the reasons they used to justify their actions and decisions.

Students had taken a sequence of three Calculus courses involving both one-variable and multi-variable functions. The project started early in the semester. Students had been introduced to differential equations through a population model and some work on anti-derivatives was done in order to remind students some Calculus concepts. No methods for solving differential equations were presented when the project was introduced. Solution of differential equations was introduced during the month they were working in their project and they could use whatever they considered could be important in their work. The model could include differential equations but this was not a requirement. In fact, initial models included algebraic representations of price versus time functions, graphs describing the price behavior with time, and only a few of them took into account variation since the beginning.

Work on the problem required of several modeling cycles. Observation guides were designed to keep trace of students' work in class and of whole class discussion. Students had to hand in their advance at the end of each cycle. All these productions were analyzed by the researcher and a teacher separately and results of the analysis were negotiated between them. Each class session was prepared by the researcher and the teacher. Discussions between them were audio recorded and all the planning sessions materials were kept for analysis. All these data, together with the transcripts form the interviews, were used to study the evolution of students' schema and the interactions between students.

Given that the goal of this paper is the analysis of students' acts of understanding and development of powerful conceptual ideas, the focus of the analysis will be the work of one group of students in a class session where they developed and analyzed their model as a case study. This analysis can shade light into some of the characteristics that may result from the use of a project where students are given some freedom of action. Results are shown according to the broad cycles students, identified as Alejandra, Carlos and Mauricio (not their real names), went through while working on the problem, with emphasis on those instances where important discussion took place in terms of the goal of this paper.

Elaboration of a model

Alejandra, Carlos and Mauricio focused their discussion on their knowledge of economics:

C: I think we can use market equilibrium here to work with supply and demand of the product. This would be my first idea here.

A: But, here we have a situation where price is not fixed, the situation is not static, it changes with time and I don't know how equilibrium can be used here. We can think of something like what we would expect, … how the prices…how we expect the prices to behave….

After some time discussing how price could change, Carlos said:

C: I think we are not sure what we are doing, I still think that equilibrium can help, but I don't know how…

M: And, if we think about equilibrium in terms of time? Something like at each time prices tend to equilibrium at that moment.

C: I don't know but then supply and demand depend on prices and prices depend on time…

A: … and expectations, for price, change according to how prices are changing and maybe… we can think of supply and demand depending on p and pe, but is pe constant?

M: No, price expectation depends on how people expect prices to change in the future, so you or the producer guess today how much it will change tomorrow, so it is like guessing if price is going up or down in the next time…. Ah, I guess it can be something like using the derivative… of p today.

A: (silence)…That makes sense, then we can have supply and demand depending on t, p of t and p prime of t.

C: Yes, and if we want something easy, we can consider supply and demand are linear functions, and in a given t, as Mau suggested we can equal them, supposing equilibrium, then…like (writes: S(p,p',q) = D(p,p',q))

A: If we do that we will have an equation that looks something as (writes) p' + ap = b after working with supply and demand, so a and b are constants.

M: Yes! That looks nice! … although I don't know how we would work with that…

The students used their function schema and their economics knowledge schema since their first encounter with the problem situation. They decided that the main variables they had to consider were price, price expectation and quantity: p, pe, q; they introduced the idea of market equilibrium, S(p,q) = D(p,q), and considered that they had to concentrate in the description of price as a function of time, consumer expectations and price expectations. They had some difficulties deciding what elements of the market situation they had to consider in order to make it possible for them to find an appropriate model. They showed evidence of having constructed a function schema with an object conception of function, and to have coordinated the function and economics schemas through the relation of prices with time and price expectation with price variation.

It is clear from their discussion that a bisociation takes place when they coordinate the two schemas pertaining to different fields of knowledge. This coordination cannot be thought of as just juxtaposing knowledge of both areas. Knowledge about static economics principles are brought to play but are not enough to handle this new situation. It is the idea that price expectation could be related to price variation which makes it possible to think of the problem in a new and creative way and new knowledge is created by this act of understanding. Not only students propose that price and price expectation can depend on time, but they create a new way to think of market equilibrium in a situation where change is involved. They also make a coordination of price expectations in their economics schema to the derivative in their function schema. This bisociation produced new knowledge: students were able to represent their new ideas in terms of a mathematical model represented by an equation involving price and price change in time which they had never studied before. In terms of schema development, we can say that the bisociation of the economics and function schemas made the emergence of a new schema, which can be called a differential equation schema (DES), probably at an Intra-DES stage characterized by the model’s mathematical representation as a differential equation, where price and price expectations where considered as functions, and price expectations were related with price change in time.

Model analysis: new tools are created

After whole group discussion where the different models used by the students were discussed, the teacher invited students to analyze them and try to find out what their proposed equation told them about the gold’s price. After a short period of time where the studied team discussed if they could solve the equation and how, without finding any specific way to do it, the following dialogue took place:

C: OK, so what can we do? Perhaps…something we could do with this (referring to the model’s equation) is… try to graph it? and…

M: But how? Well yes. Since p’ depends only on p, we can plot that function.

A: It is easy to make a graph since it is a line… (draws a line like the one shown in the next figure 1)

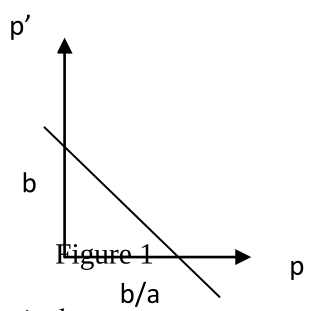

Figure 1

A: Now… what I see is that p’ decreases as p increases, and when p is b/a the derivative is zero.

C: No, that is not what happens there, what the graph shows is p’ is positive when p is between zero and b/a, so there it increases… after that it decreases.

A: So it should have a maximum at b/a, but I don’t know. See, if p’= 0, p=b/a, it’s true, but…what happens? b/a is a constant… what we need to know is how the price changes with time, so…

C: Isn’t it the other way round?.... Oh no!...I see. Ah, you’re right, I see it! When p’ is zero, it is zero for all the time, because p’ is a function of t and p is also a function of t… so the price does not change.

M: This is like equilibrium, isnt’ it? Then p’ is positive for p under b/a, so if at the beginning the price is under b/a in time, the price will decrease and if it is above, it will increase… Something like (figure 2)

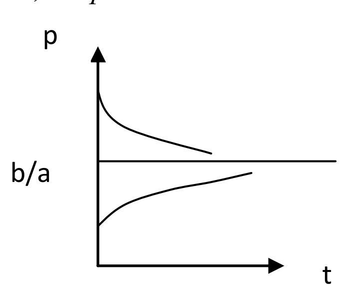

Figure 2

C: Seems that they tend to b/a, would they reach that line?

A: I don't know, but what it seems is that in the long term the price tends to the equilibrium. It makes sense in economy….This is great!

After developing a mathematical model in terms of a new type of equation, students worried about how to interpret the equation, how to solve it and if there was a way to test if what they were doing was useful in understanding and solving the initial problem. Again, through their discussion, they bring together their previous knowledge about graphs of functions and use it to create a new representation for the equation they are working with. A new bisociation occurs here as students bring together their function and derivative schemas which results in a new graphical representation they had never used before. This bisociation is made possible by the need of students to interpret their model in terms of what could be expected of the behavior of prices with time. In doing so, they discover a new tool to interpret their mathematical model, the phase plane representation of an autonomous differential equation, which had not been introduced before. The development of new knowledge is new act of understanding is reflected by a clear understanding of derivative as a transformation of the original function, and students' interpretation of the graph in terms of the derivative's properties and the behavior of the price function with time as shown in figure #2. It is clear, from APOS point of view, that students developed through this analysis an Inter-DES where prices and price expectations are considered as functions, price expectation is considered as a transformation of prices and both are related through the phase space diagram from which new information can be obtained and described graphically.

In whole group discussion, the teacher asked: *can you explain your diagram?* They respond

A: *We drew it to see how the relationship between p and p' can go, and it turned out that it can explain what happens in the long term without doing anything to the equation.*

M: *Yes, you can see what happens if p' changes, no, the other way round, p' changes with p…*

And they explained their results.

Teacher: *Why linear?*

M: *Because the easier form for the demand and supply functions is to suppose they are linear, and we also supposed that they could be equal for every time, although we don't know if that is possible…*

This episode gives evidence of students' new DES-schema development. This development resulted from their creative linking of different schemas when dealing with a novel situation. In Koestler terms, they perceived an idea in two self- consistent but habitually incompatible frames of reference, and from that bisociation new understanding of differential equations emerged. These students were able to develop new tools to interpret equations of this type and to relate this new knowledge with their previous knowledge.

Another interesting discussion took place near the end of the class session:

M: Well the graph looks like an exponential, well two… it depends on where you are at the start of time, and you see? We have p'+ ap =b, or p' = b-ap, can we integrate? But how?

C: I think I now know… you do p'/b –ap and integrate it is a logarithm, but what is at the other side of the equal sign?

A: You have 1, so you integrate and, that is p? no p is function of t so it should be t. And if we rewrite it we have an exponential function, so it does not intersect the line b/a.

Finally while using the DES- schema, students were able to construct new relations, this time between the graphical representation of the solution obtained from the phase plane and the development of a solution method for the equation of the model. This new bisociation of the function schema with the integration of function's schema resulted in finding a way to analytically solve the differential equation. In doing this, a new development of their DES-schema occurred through the introduction of the integral function, a component of the function schema to the DES-schema through an assimilation mechanism.

The whole episode shows that the description of the modeling process using APOS theory makes sense. Through their work with the modeling problem, students created a new schema where function and economics schemas were no longer two separate entities. They freely developed relations among the components of this new schema that promoted its development. Students became aware of the possibilities this new schema offered in terms of interpretation of the proposed model and that, through actions applied to the differential equation, new tools and concepts could emerge. This episode is an example of the power of constructive generalization in the development of new knowledge where creativity manifested through bisociations is evidenced.

Conclusions

These students showed to have coordinated two previously constructed schemas: the static economics schema and the function derivative schema which included derivative and integral as objects. By considering elements of their economics schema, such as equilibrium, supply, and demand in terms of price, they constructed a relation between price and time and also coordinated the notion of price expectation with that of derivative. This coordination made it possible to find a mathematical model in terms of a differential equation and the emergence of a new schema which made possible the creation of new knowledge and the construction of a differential equation as an object that could be represented in a graph and, through actions, rewritten in a form where integration was possible. None of the developed tools had been introduced in the course, they were created by these students in the classroom.

The results of this study show that when students are given the opportunity to work by themselves in an interesting problem, they are able to use their previous knowledge in new and creative ways. Students showed that they used their schemas to construct a novel representation of the differential equation representing the model. They could also use it to use it with economics concepts as a guide in order to verify if the way they were thinking made sense in terms of the problem they wanted to solve, and developed a way to find a solution to the equation. The coordination between these two schemas resulted in the creation of a representation of the equation that could be used as an object of reflection about the behavior of prices. The coordination of these schemas enabled the emergence of a powerful conceptual tool to be able to construct arguments to validate students' thinking. It also served as a means to think on the mathematical form of the solution that was coordinated with their knowledge of integration of real functions to find a way to solve the differential equation describing the model.

This study also shows how creativity play a role in students' learning when adequate conditions and adequate problem situations are available in their working environment. Moreover, APOS theory and Koestler's theory of creativity served as useful tools to better understand student's work in terms of the emergence and development of a new schema and in terms of bisociations which made this emergence and learning possible.

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